# Algebraic structures on moduli of curves from vertex operator algebras

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The moduli stack $\overline{\mathcal{M}}_{g,n}$, parametrizing families of Deligne-Mumford stable $n$-pointed curves of genus $g$, has proved crucial for the study of smooth curves and their degenerations. Vertex operator algebras (VOAs) have played a similar role in the study of conformal field theories, finite group theory, the construction of knot invariants, and 3-manifold invariants. By fixing coordinates at marked points, representations of VOAs may be used to study $\overline{\mathcal{M}}_{g,n}$. Namely, by associating to every stable $n$-pointed and coordinitized curve, a collection of $V$-modules, one can construct a vector space of coinvariants, the largest quotient of the tensor product of the modules on which a natural Lie algebra acts trivially. Varying these data gives sheaves on $\widehat{\mathcal{M}}_{g,n}$, the space parametrizing families of stable n-pointed coordinatized curves. Changing coordinates gives $\widehat{\mathcal{M}}_{g,n}$ the structure of a torsor over $\overline{\mathcal{M}}_{g,n}$. Assumptions on modules ensure sheaves descend to $\overline{\mathcal{M}}_{g,n}$.

After work in mathematical physics, the first examples of sheaves of coinvariants studied in algebraic geometry came from certain affine VOAs, derived from a simple Lie algebra $\mathfrak{g}$, and a positive integer $\ell$. By [TK87, TUY89] these give vector bundles that support a projectively flat connection, and satisfy factorization, allowing ranks to be computed recursively. Analogous results were shown to hold for sheaves from (discrete series) Virasoro VOAs [BFM91], more general VOAs on smooth pointed coordinitized curves [FBZ04], and for VOAs with finite and semisimplicity properties on rational curves [NT05]. For examples given by affine VOAs more was shown: Chern characters form cohomological field theories and Chern classes lie in the tautological ring [MOP15, MOP$^+$17]. If $g = 0$, they are globally generated [Fak12]. There are canonical isomorphisms of (dual) fibers with generalized theta functions [Ber93, Tha94, BL94, Fal94, Pau96, LS97, BG19, BF19].

Much of this theory has recently been extended: Sheaves defined by VOAs of CFT-type are quasi-coherent, carrying a projectively flat connection with logarithmic singularities on the boundary of $\widehat{\mathcal{M}}_{g,n}$ [DGT21]. If $V$ is $C_2$-cofinite and rational, these satisfy factorization, and are vector bundles [DGT22a]. If also simple and self-dual, Chern characters form cohomological field theories, and Chern classes are tautological [DGT22b]. If $V$ is generated in degree 1, and $g = 0$, are globally generated [DG23]. Many examples of $C_2$-cofinite and rational VOAs giving such vector bundles are known [DGT22a, §9], [DGT22b, §5], [DG23, §7-9].

Sheaves of coinvariants are known to be coherent in various cases [DGK22], [DG23]. As vector bundles are critical to the study of moduli, it is natural to ask:

**Question.** *When are coherent sheaves of coinvariants locally free?*

To answer the question, in [DGK23] we define a series of associative algebras $\mathfrak{A}_d(V)$ for $d \in \mathbb{Z}_{\geq 0}$. To describe these and our main geometric application, [DGK23, Corollary 5.2.6] (stated below) we start with a small amount of notation.

0.1. **Background: VOAs, and V-modules, and sheaves of coinvariants.** The input used to define a VOA includes a vector space, a vertex operator, together with the vacuum and conformal vectors. These satisfy a long list of axioms (see eg. [DGK23, §1]). We consider VOAs satisfying finiteness assumptions $C_1$-cofiniteness, and the stronger $C_2$-cofiniteness, as well as rationality. By [Liu22], a VOA is $C_1$ if and only if it is (strongly) finitely generated. A VOA is said to be rational if every finitely generated module is a finite sum of simple modules.

There is a one-to one correspondence between simple $V$-modules and simple modules over $A(V)$, an associative algebra constructed as a quotient of $V$. Important properties of $V$ are reflected in $A(V)$. For instance, if $V$ is $C_1$-cofinite, then $A(V)$ is finitely generated, If $V$ is $C_2$-cofinite, then $A(V)$ is finite dimensional, and if $V$ is rational, then $A(V)$ is finite and semi-simple (for references, see [DGK23]).

As in [DGK23, §2], one may construct (left and right) Verma modules for $V$ via a normed, associative algebra $\mathscr{U} = \mathscr{U}(V)$ as follows. A left $\mathsf{A}$-module $W_0$ is a module over a subring $\mathcal{U}_0 \subset \mathcal{U}$. We set $\Phi^{\mathsf{L}}(W_0) = \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right) \otimes_{\mathscr{U}_0} W_0 = \oplus_d \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right)_d \otimes_{\mathscr{U}_0} W_0$, an $\mathbb{N}$-graded module. Given a right $\mathsf{A}$-module $Z_0$, we set $\Phi^{\mathsf{R}}(Z_0) = Z_0 \otimes_{\mathscr{U}_0} \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right)$. Here $\mathrm{N}^1_{\mathsf{L}}\mathscr{U}$ and $\mathrm{N}^1_{\mathsf{R}}\mathscr{U} \subset \mathscr{U}$ are neighborhoods of 0.

To describe the sheaf of coinvariants, let $\pi \colon \mathscr{C} \to S$ be a projective curve, with $n$ distinct smooth sections $P_i \colon S \to \mathscr{C}$ and formal coordinates $t_i$ at $P_i$, and let $W^\bullet = W^1 \otimes \cdots \otimes W^n$ be the tensor product of an $n$-tuple of $V$-modules. We set $\mathscr{W} := W^\bullet \otimes \mathcal{O}_S$, and $\mathscr{L} := \mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ the sheaf of Chiral Lie algebras [DGT21, DGT22a], explicitly described in [DGK23, §4.4]. The *sheaf of coinvariants* $[\mathscr{W}]_{\mathscr{L}}$ on $S$ is defined to be the cokernel $\mathscr{L} \otimes_{\mathscr{O}_S} \mathscr{W} \to \mathscr{W} \to [\mathscr{W}]_{\mathscr{L}} \to 0$.

0.2. **Mode transition algebras.** The mode transition algebra, is defined as $\mathfrak{A} = \Phi^{\mathsf{R}}(\Phi^{\mathsf{L}}(\mathsf{A})) = \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(\mathsf{A})) = \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right) \otimes_{\mathscr{U}_0} \mathsf{A} \otimes_{\mathscr{U}_0} \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right)$, a bigraded vector space. By [DGK23, Def. 3.2.5] $\mathfrak{A}$ has an algebra structure, and $\Phi^{\mathsf{L}}(W_0) = \oplus_d \Phi^{\mathsf{L}}(W_0)_d$ is a left $\mathfrak{A}$-module. The subalgebra $\mathfrak{A}_d := \mathfrak{A}_{d,-d}$ of $\mathfrak{A}$ acts on $\Phi^{\mathsf{L}}(W_0)_d$.

0.3. **Geometric application.** Mode transition algebras reflect both the algebraic structure of $V$ and the geometry of coinvariants. On the geometric side, we show coherent sheaves of coinvariants are locally free when the mode transition algebras admit unities that act as identities on modules (we call these strong unities).

A sheaf of $\mathcal{O}_S$-modules is locally free if and only if is coherent and flat. So when $[\mathscr{W}]_{\mathscr{L}}$ is coherent, to show it is locally free, it suffices to show it is flat. For this one may reduce to showing that vector spaces of coinvariants have the same dimension over all pointed and coordinatized curves.

The standard approach, from [TUY89, Theorem 6.2.1], relies on the factorization property to argue that this holds. However, since we do not assume $V$ is rational or $C_2$-cofinite, and so particular, $A(V)$ may not be finite dimensional or semi-simple, by [DGK22, Proposition 7.1], one cannot apply factorization. Instead, by smoothing (as described in [DGK23]), we show that if the mode transition algebras admit identity elements satisfying any of the equivalent properties of [DGK23, Definition/Lemma 3.3.1], then one can identify the rank of the space of coinvariants at a nodal curve at the central fiber of a degeneration with the rank of the coinvariants on the general fiber. This allows one to argue inductively, reducing the number of nodes. As a consequence of [DGK23, Theorem 5.0.3], if strong unities exist, so this smoothing process for $V$ may be carried out, we show:

**Corollary.** [DGK23, Corollary 5.2.6] *Let $W^1, \ldots, W^n$ be simple modules over a $C_1$-cofinite vertex operator algebra $V$, such that coinvariants are coherent for curves of genus $g$, and such that $\mathfrak{A}_d(V)$ admit strong unities for all $d \in \mathbb{Z}_{\geq 0}$. Then sheaves of coinvariants are locally free. If the conformal dimensions of $W^1, \ldots, W^n$ are rational, these sheaves descend to vector bundles on $\overline{\mathcal{M}}_{g,n}$.*

*Proof.* (sketch) To prove the claim, we induct on the number of nodes via degenerations. The base case follows from the assumption of coherence and the fact that sheaves of coinvariants support a projectively flat connection on schemes parametrizing families of smooth curves [FBZ04, DGT21]. □

0.4. **Examples.** One has $\mathfrak{A}_0 = \mathsf{A}$, which has a unity [DGK23, Remark 3.2.4]. By [DGK23, Example 3.3.2] if $V$ is rational and $C_2$-cofinite, then $\mathfrak{A}_d$ admit strong unities for all $d \in \mathbb{N}$, and the Corollary specializes to [DGT22a, VB Corollary]. By [DGK23, Proposition 7.2.1], the Heisenberg VOA (which is $C_1$-cofinite, but neither $C_2$-cofinite, nor rational) also admits strong unities for all $d \in \mathbb{N}$. In particular, by the Corollary, associated sheaves of coinvariants which are coherent if $g = 0$ are (globally generated) vector bundles [DGK23, Corollary 7.4.1].

0.5. **Questions.** There are a number of questions about these sheaves, as asked in [DGT22b, §6], for $V$ of CohFT-type, and in [DG23, §10] for $V$ generated in degree 1. One may also ask: Can one find Chern classes of associated bundles (in this more general setting where $V$ is not of CohFT-type) and determine when the higher Chern classes are tautological? What are sufficient conditions ensuring the Chern classes are effective? Can one realize the infinitely many extremal rays of the effective cone known to exist from [CLTU23] as first Chern classes of bundles of coinvariants? There are natural maps between different sheaves of coinvariants; what can one say about their degeneracy classes or loci? Finally, as asked in [DGT22b] for $V$ of CohFT-type, and in [DG23] for $V$ is generated in degree 1, are there geometric interpretations of (dual) fibers in the case where $V$ is $C_1$-cofinite (and so strongly finitely generated)? For instance, for lattice VOAs $V_L$, where $L'/L \cong Z/mZ$, and $m = 2k$, which is generated in degree $k$, Ueno and Nagatomo [Uen95] gave an identification with theta functions ($V_L$ are called lattice Heisenberg VOAs in [BD04, §4.9]). Dimensions of spaces of conformal blocks are computed as an application (see [DGT22b, Ex 5.2.5] for an alternative approach).

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