ON FACTORIZATION AND VECTOR BUNDLES OF CONFORMAL BLOCKS FROM VERTEX ALGEBRAS

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Abstract. Modules over vertex operator algebras give rise to sheaves of coinvariants and conformal blocks on moduli of stable pointed curves. Here we prove the factorization conjecture for these sheaves. Our results apply in arbitrary genus and for a large class of vertex algebras. As an application, sheaves defined by finitely generated admissible modules over vertex algebras satisfying natural hypotheses are shown to be vector bundles. Factorization is essential to a recursive formulation of invariants, like ranks and Chern classes, and to produce new constructions of rational conformal field theories and cohomological field theories.

By assigning a module over a vertex operator algebra to each marked point on a stable pointed curve, one can construct dual vector spaces of coinvariants and conformal blocks. These give rise to sheaves on the moduli space of stable pointed curves [DGT1]. We prove here, under natural assumptions, the Factorization Theorem, showing that vector spaces of coinvariants and conformal blocks satisfy a twenty-five-year-old conjecture [Zhu1, FBZ]. In particular, at nodal curves, they are decomposable into analogous objects defined on curves of smaller genus.

More generally, we show that sheaves of coinvariants at infinitesimal smoothings of nodal curves satisfy the Sewing Theorem (8.3.1), a refined version of the Factorization Theorem. Using this and their projectively flat connection [DGT1], we show that these sheaves are vector bundles (VB Corollary). Such bundles are of interest for a number of reasons.

Conformal field theories describe classical solutions of the string equations of motion. Rational conformal field theories arise from rigid nondegenerate modular tensor categories constructed from vector bundles on moduli spaces of curves satisfying factorization [MS, Seg1, Seg2]. Examples were known in genus zero and one [Hua4], as well as for vertex algebras induced from affine Lie algebras and the Virasoro Lie algebra [TUY, BFM]. Our results give new modular functors, and indicate the possibility of obtaining rational conformal field theories for many vertex algebras in all genera. Examples include bundles defined from modules over even lattice vertex algebras, certain simple W-algebras, orbifold algebras and commutants (like parafermions and code VOAs), as well as tensor products of these (see §9 for more details).

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Factorization leads to recursions and allows one to make inductive arguments. The ranks of the vector bundles we construct can therefore be recursively computed, as was done for bundles defined by integrable modules at a fixed level over affine Lie algebras (see [Bea, Sor] for an account of the Verlinde formula). Factorization is used to show that Chern characters give rise to *semisimple cohomological field theories* when the vector bundles are defined by self-contragredient vertex algebras, as we prove in [DGT2], building on the results known for affine Lie algebras [MOP1, MOP$^+$2]. Consequently, Chern classes can be recursively computed, and lie in the tautological ring of the moduli space of stable curves. Having at our disposal new vector bundles with tautological Chern classes, could potentially lead to some progress on open questions on the tautological ring of moduli spaces of curves of positive genus. For instance, as proposed by Pandharipande [Pan], an approach to the computation of such Chern classes independent of the projective flatness of their connections [DGT1] would yield relations in the tautological ring, and this could be used to test Pixton's conjecture on the set of tautological relations [Pix, Jan].

In the special case when defined by integrable modules over affine Lie algebras, vector spaces of conformal blocks are canonically isomorphic to generalized theta functions [BS, Ber, Tha, BL1, Fal, Pau, LS]. When in addition the genus is zero, vector bundles of such coinvariants are globally generated [Fak], and so their Chern classes have positivity properties. For instance, first Chern classes are base-point-free, and hence give rise to morphisms, some with images known to have modular interpretations [GS, GG, Gia, BG, GJMS]. It is natural to expect that the more general vector bundles of coinvariants and conformal blocks studied here can be shown to have analogous properties under appropriate assumptions.

To state our results, we briefly set a small amount of notation. Let $(C, P_\bullet)$ be a stable n-pointed curve with $P_\bullet = (P_1, \dots, P_n)$, and let $M^1, \dots, M^n$ be finitely generated admissible modules over a vertex operator algebra V (see §§1.1-1.2 for some background). When V is rational, such modules are finite sums of simple admissible V-modules. We make use of the correspondence between isomorphism classes of simple V-modules and finite-dimensional simple modules over Zhu's associative algebra $A(V)$ [Zhu2] (§1.6).

The vector space of coinvariants $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)}$ is the largest quotient of the tensor product $\otimes_{i=1}^n M^i$ by the action of a Lie algebra determined by V and $(C, P_\bullet)$. Two Lie algebras have been used to define coinvariants: Zhu's Lie algebra $\mathfrak{g}_{C\setminus P_\bullet}(V)$, and the chiral Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$.

Zhu's Lie algebra (§A.1) is defined when the vertex algebra V is quasi-primary generated and $\mathbb{Z}_{\geq 0}$-graded with lowest degree space of dimension one, for either *fixed* smooth curves [Zhu1, AN], or for *rational* stable pointed curves *with coordinates* [NT]. To show that $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is a Lie algebra, Zhu uses that any *fixed* smooth algebraic curve admits an atlas such that all transition functions are Möbius transformations. Transition functions between charts on families of curves of arbitrary genus are more complicated, and for an arbitrary vertex operator algebra, $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is not well-defined.

The advantage of the Chiral Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$ is that it is available for curves of arbitrary genus, and for any vertex operator algebra V, not necessarily quasi-primary generated. Based on work of Beilinson, Feigin, and Mazur for Virasoro algebras [FBZ], the chiral Lie algebra was defined for smooth pointed curves by Frenkel and Ben-Zvi [FBZ, §19.4.14], and shown in [FBZ] to coincide with that studied by Beilinson and Drinfeld [BD]. In [DGT1], we extended the definition of $\mathscr{L}_{C\setminus P_\bullet}(V)$ to nodal curves, making it possible to define coinvariants on stable pointed curves of arbitrary genus by action of the chiral Lie algebra. Here in §3 we very slightly modify $\mathscr{L}_{C\setminus P_\bullet}(V)$, to define a new chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$, whose coinvariants share all of the features of the coinvariants defined in [DGT1], and as we show here, additionally satisfy factorization. A detailed description of $\mathcal{L}_{C\setminus P_\bullet}(V)$ on nodal curves is given in §3.3.

In [NT], Nagatomo and Tsuchiya remark that the coinvariants on rational curves with coordinates using Zhu's Lie algebra are equivalent to those considered by Beilinson and Drinfeld. In Proposition A.2.1, we verify their genus zero statement, and further show that coinvariants from the chiral Lie algebra are isomorphic to those given by $\mathfrak{g}_{C\setminus P_\bullet}(V)$ whenever Zhu's Lie algebra is defined. For instance, when V is quasi-primary generated and one works over a family of curves admitting an atlas where all transition functions are Möbius transformations, both perspectives are equivalent. This perhaps surprising result gives the first hint for why Zhu's conjecture can be extended to *coordinate-free* curves of arbitrary genus using a different construction, and in principal for perhaps more general vertex algebras.

Let C be a stable curve with one node Q, let $\widetilde{C}\to C$ be the normalization, $Q_+, Q_-\in\widetilde{C}$ the preimages of Q, and set $Q_\bullet=(Q_+,Q_-)$. Let $\mathscr{W}$ be the set of simple admissible V-modules, and for $W\in\mathscr{W}$, let W' be its contragredient module (§1.8). Our main result is:

Factorization Theorem. *Let $V=\oplus_{i\geq 0}V_i$ be a rational and C_2-cofinite vertex operator algebra with $V_0\cong\mathbb{C}$. One has:*

$$\mathbb{V}(V;M^\bullet)_{(C,P_\bullet)}\cong\bigoplus_{W\in\mathscr{W}}\mathbb{V}\left(V;M^\bullet\otimes W\otimes W'\right)_{\left(\widetilde{C},P_\bullet\sqcup Q_\bullet\right)}.$$

It may be that $\widetilde{C}=C_+\sqcup C_-$ is disconnected, with $Q_\pm\in C_\pm$. Then:

$$\mathbb{V}\left(V;M^\bullet\otimes W\otimes W'\right)_{(\widetilde{C},P_\bullet\sqcup Q_\bullet)}\cong\mathbb{V}\left(V;M_+^\bullet\otimes W\right)_{X_+}\otimes\mathbb{V}\left(V;M_-^\bullet\otimes W'\right)_{X_-}$$

where $X_\pm=(C_\pm,P_\bullet|_{C_\pm}\sqcup Q_\pm)$, and $M_\pm^\bullet$ are the modules at the $P_\bullet$ on $C_\pm$.

The isomorphism giving the Factorization Theorem is constructed in §7.

The results presented here generalize work of Nagatomo and Tsuchiya for coinvariants defined using Zhu's Lie algebra on stable coordinatized curves of genus zero [NT], of Huang in genus zero and one for coinvariants defined by $\mathbb{N}$-graded weak modules over certain self-contragredient vertex algebras

[Hua1, Hua2, Hua4, Hua5], of Beilinson, Feigin, and Mazur on coinvariants defined from Virasoro algebras [BFM], as well as coinvariants defined by Tsuchiya, Ueno, and Yamada [TUY] from integrable modules over affine Lie algebras. A brief history is given in §0.1.

With similarities to special cases, our proof has fundamental differences. As described, to allow for families of higher genus curves and vertex algebras satisfying only mild constraints, we construct coinvariants using the more general chiral Lie algebra. While the coinvariants defined with the chiral Lie algebra are isomorphic to those defined from Zhu's Lie algebra when they are both defined (Proposition A.2.1), working on the general setting considered here requires new ideas and approaches. To our advantage, the chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ has an algebro-geometric formulation, as sections on $C \setminus P_\bullet$ of a sheaf of Lie algebras determined by the vertex algebra V. Moreover, the category of vertex algebras in which we work is very well-studied. In particular, techniques from algebraic geometry and a number of useful results about finitely generated admissible modules over such vertex algebras are available, emphasizing the naturality of our assumptions, and perhaps explaining why earlier findings had not been further extended.

Propositions 3.3.1, 5.1.1, and 6.2.1 are at the heart of this work, and arise from the construction in §2 of the sheaf $\mathcal{V}_C$ globalizing a vertex algebra V over a nodal curve C, integral to the definition of the chiral Lie algebra. The sheaf $\mathcal{V}_C$ is a slight variant on the sheaf $\mathscr{V}_C$ that we defined in [DGT1]. The two sheaves coincide for smooth curves, and as we explain here, we recover the results of [DGT1] using $\mathcal{V}_C$ in place of $\mathscr{V}_C$. In particular, while the projectively flat connection can be obtained with two different constructions, to have factorization, one has to use $\mathcal{V}_C$ (see §2.5, §2.6, and §2.7).

In Proposition 3.3.1, we explicitly describe the chiral Lie algebra on a nodal curve in terms of elements of the chiral Lie algebra on its normalization. This involves stable k-differentials that satisfy an infinite number of identities. To study coinvariants, a module Z (defined in §6.2) comes into play in the normalization of the curve, when the attaching points of a node are untethered. To work with Z, we require that V is rational. This implies that, in addition to being finite, $A(V)$ is semisimple (§1.7), giving with some finesse, the decomposition of coinvariants as in the Factorization Theorem.

Proposition 6.2.1 allows us to reinterpret the (decomposed) coinvariants on the normalization as coinvariants on the nodal curve. Key to our argument is that the coinvariants we consider are finite-dimensional for smooth curves (Proposition 5.1.1), and so we may study them via their dual spaces, the spaces of conformal blocks, using correlation functions (Remark 4.2.1).

In Proposition 5.1.1 we show vector spaces of coinvariants defined from the chiral Lie algebra and smooth curves of arbitrary genus are finite-dimensional. Known to be true in special cases, this result is a natural generalization of work of Abe and Nagatomo [AN] for coinvariants defined from Zhu's Lie algebra and smooth curves with formal coordinates (see §0.2 for history of the problem). As in [AN], for this result we assume V is C_2-cofinite, that is, a certain subspace of V has finite codimension. This property also implies that V has finitely many simple modules.

As an application of the Factorization Theorem, we show:

VB Corollary. *Let $V = \oplus_{i\geq 0} V_i$ be a rational, C_2-cofinite, vertex operator algebra with $V_0 \cong \mathbb{C}$, and $M^\bullet$ an n-tuple of finitely-generated admissible V-modules. Then $\mathbb{V}(V; M^\bullet)$ is a vector bundle of finite rank on $\overline{\mathcal{M}}_{g,n}$.*

The proof of VB Corollary is presented in §8. The result on the interior of $\overline{\mathcal{M}}_{g,n}$, the moduli space $\mathcal{M}_{g,n}$ of smooth pointed curves, follows from finite-dimensionality of coinvariants (Proposition 5.1.1) and the existence of a projectively flat connection [DGT1]. Two ingredients are needed to give VB Corollary on the whole space $\overline{\mathcal{M}}_{g,n}$: Theorem 8.2.1, a more general result on finite-dimensionality of coinvariants, and the Sewing Theorem (8.3.1). Each of these involve evaluation of the sheaf of coinvariants on a family formed by infinitesimally smoothing a nodal curve (§8.1).

The proof of the Sewing Theorem (§8) relies on the Factorization Theorem and a sewing procedure originally found in [TUY, §6.2]. We can further interpret the Sewing Theorem as a decomposition induced by a differential operator (Remark 8.3.3) which naturally arises from the twisted logarithmic $\mathcal{D}$-module structure of sheaves of coinvariants [DGT1, §7].

0.1. **History of factorization.** Tsuchiya and Kanie used integrable modules at a fixed level over affine Lie algebras to form sheaves of coinvariants on moduli of smooth pointed rational curves with coordinates [TK]. Generalized by Tsuchiya, Ueno, and Yamada to moduli of stable pointed coordinatized curves of arbitrary genus, these sheaves were shown to be vector bundles and to satisfy a number of good properties including factorization [TUY]. Tsuchimoto [Tsu] proved the bundles are independent of coordinates and descend to $\overline{\mathcal{M}}_{g,n}$. Beilinson, Feigin, and Mazur [BFM] showed that factorization holds for coinvariants defined by modules over Virasoro algebras. Our arguments follow [NT] in the genus zero case after our study of the chiral Lie algebra allows one to replace Zhu's Lie algebra in the general case.

Factorization was proved for $g \in \{0, 1\}$ by Huang [Hua1, Hua2, Hua4, Hua5]. His approach was to prove the operator product expansion and the modular invariance of intertwining operators, the two conjectures that Moore and Seiberg formulated and used to derive their polynomial equations [MS]. To construct genus zero and genus one chiral rational conformal field theories, Huang assumes that (i) $V = \oplus_{i\geq 0} V_i$ with $V_0 \cong \mathbb{C}$; (ii) Every $\mathbb{N}$-gradable weak V-module is completely reducible; and (iii) V is C_2-cofinite. Our assumptions are (1) $V = \oplus_{i\geq 0} V_i$ with $V_0 \cong \mathbb{C}$; (2) V is rational; and (3) V is C_2-cofinite. Given (i) = (1), conditions (ii) and (iii) of Huang are equivalent to our conditions (2) and (3) (see §1.7 for more details). Huang shows that if one assumes in addition (iv) $V \cong V'$, then the modular tensor categories he constructs for $g \in \{0, 1\}$ are *rigid* and *nondegenerate*.

Codogni in [Cod] proves factorization in case our hypotheses hold, with the additional assumptions (iv) $V \cong V'$ and (v) V has no nontrivial modules. Like Nagatomo and Tsuchiya, Codogni works with coinvariants defined on the moduli space of curves with formal coordinates.

We note that the additional assumption (iv) gives that V is quasi-primary generated and in particular, coinvariants defined from Zhu's Lie algebra and the chiral Lie algebra are both well-defined and agree (see §A).

Here we do not assume condition (iv) nor condition (v).

We do not know of any vertex algebra that satisfies our conditions (1–3) but not condition (iv). Therefore, while having new results for $g \geq 2$ with many examples, we cannot say whether this work covers any new example for $g \in \{0, 1\}$. The added value in those special cases perhaps lies in the difference of perspectives. After all, a complete classification of vertex algebras satisfying conditions (1–3) is yet elusive. On the contrary, there exist plenty of examples satisfying (1–3) but not (v), see §9.

0.2. **Finite-dimensionality of coinvariants: prior speculations and results.** In [FBZ, page 3 and §5.5.4] the authors single out rational vertex algebras as good candidates for defining finite-dimensional coinvariants (and hence leading to finite-rank vector bundles). Interestingly, it seems that rationality and C_2-cofiniteness were believed to be equivalent [DLM2, ABD] for vertex algebras at that time (all known cases of vertex algebras having either one of the properties, in fact had both). This has been disproved, as Abe has given a C_2-cofinite, non-rational vertex operator algebra [Abe].

Finite-dimensionality of coinvariants has previously been proved in the following special cases: (1) for integrable modules at a fixed level over affine Lie algebras [TUY, Thm 4.2.4]; (2) modules over C_2-cofinite Virasoro vertex algebras [BFM, §7]; (3) curves of genus zero and modules over C_2-cofinite vertex operator algebras $V = \oplus_{i\in\mathbb{N}} V_i$ such that $V_0 \cong \mathbb{C}$ [NT, Thm 6.2.1]; (4) fixed smooth curves of positive genus and modules over a quasi-primary generated, C_2-cofinite vertex operator algebras $V = \oplus_{i\in\mathbb{N}} V_i$ such that $V_0 \cong \mathbb{C}$ [AN, Thm 4.7].

1. Background

1.1. **Vertex operator algebras.** A *vertex operator algebra* is a 4-tuple $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$, throughout simply denoted V for short, such that:

(i) $V = \oplus_{i\in\mathbb{Z}_{\geq 0}} V_i$ is a vector space over $\mathbb{C}$ with $\dim V_i < \infty$;
(ii) $\mathbf{1}^V \in V_0$ (the *vacuum vector*), and $\omega \in V_2$ (the *conformal vector*);
(iii) $Y(\cdot, z)\colon V \to \mathrm{End}(V)\left[\!\left[z, z^{-1}\right]\!\right]$ is a linear function assigning to every element $A \in V$ the *vertex operator* $Y(A, z) := \sum_{i\in\mathbb{Z}} A_{(i)} z^{-i-1}$.

The datum $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$ must satisfy the following axioms:

(a) *(vertex operators are fields)* for all $A, B \in V$, $A_{(i)}B = 0$, for $i >> 0$;

(b) *(vertex operators of the vacuum)* $Y(\mathbf{1}^V, z) = \mathrm{id}_V$:

$$\mathbf{1}^V_{(-1)} = \mathrm{id}_V \qquad \text{and} \qquad \mathbf{1}^V_{(i)} = 0, \quad \text{for } i \neq -1,$$

and for all $A \in V$, $Y(A,z)\mathbf{1}^V \in A + zV[\![z]\!]$:

$$A_{(-1)}\mathbf{1}^V = A \qquad \text{and} \qquad A_{(i)}\mathbf{1}^V = 0, \qquad \text{for } i \geq 0;$$

(c) *(weak commutativity)* for all $A, B \in V$, there exists an $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^N \left[Y(A,z), Y(B,w)\right] = 0 \quad \text{in } \mathrm{End}(V)\left[\!\left[z^{\pm 1}, w^{\pm 1}\right]\!\right];$$

(d) *(conformal structure)* for $Y(\omega, z) = \sum_{i\in\mathbb{Z}} \omega_{(i)} z^{-i-1}$,

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p-q)\,\omega_{(p+q+1)} + \frac{c}{12}\,\delta_{p+q,0}\,(p^3-p)\,\mathrm{id}_V.$$

Here $c \in \mathbb{C}$ is the *central charge* of V. Moreover:

$$\omega_{(1)}|_{V_i} = i \cdot \mathrm{id}_V, \quad \text{for all } i, \qquad \text{and} \qquad Y\left(\omega_{(0)}A, z\right) = \partial_z Y(A,z).$$

1.1.1. *Action of Virasoro.* The conformal structure encodes an action of the Virasoro (Lie) algebra Vir on V. To explain, we recall the definition.

The *Witt (Lie) algebra* $\mathrm{Der}\,\mathcal{K}$ represents the functor which assigns to a $\mathbb{C}$-algebra R the Lie algebra $\mathrm{Der}\,\mathcal{K}(R) := R(\!(z)\!)\partial_z$ generated over R by the derivations $L_p := -z^{p+1}\partial_z$, for $p \in \mathbb{Z}$, with relations $[L_p, L_q] = (p-q)L_{p+q}$. The *Virasoro (Lie) algebra* Vir represents the functor which assigns to R the Lie algebra generated over R by a formal vector K and the elements L_p, for $p \in \mathbb{Z}$, with Lie bracket given by

$$[K, L_p] = 0, \qquad [L_p, L_q] = (p-q)L_{p+q} + \frac{K}{12}(p^3-p)\delta_{p+q,0}.$$

A representation of Vir has *central charge* $c \in \mathbb{C}$ if $K \in \mathrm{Vir}$ acts as $c{\cdot}\mathrm{id}$. By making the identification $L_p = \omega_{(p+1)} \in \mathrm{End}(V)$, Axiom (d) above implies that Vir acts on V with central charge c.

1.1.2. *Degree of $A_{(i)}$.* As a consequence of the axioms, one has $A_{(i)}V_k \subseteq V_{k+d-i-1}$ for homogeneous $A \in V_d$ (see e.g., [Zhu2]). We will then say that the degree of the operator $A_{(i)}$ is

$$\deg A_{(i)} := \deg(A) - i - 1, \qquad \text{for homogeneous } A \text{ in } V. \tag{1}$$

Axiom (d) implies that L_0 acts as a *degree* operator on V, and L_{-1}, called the *translation* operator, is determined by $L_{-1}A = A_{(-2)}\mathbf{1}^V$, for $A \in V$.

1.2. **Modules of vertex operator algebras.** Let $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$ be a vertex operator algebra. A *weak V-module* is a pair $\left(M, Y^M(\cdot, z)\right)$, where:

(i) M is a vector space over $\mathbb{C}$;

(ii) $Y^M(\cdot, z)\colon V \to \mathrm{End}(M)\left[\!\left[z, z^{-1}\right]\!\right]$ is a linear function that assigns to $A \in V$ an $\mathrm{End}(M)$-valued *vertex operator* $Y^M(A,z) := \sum_{i\in\mathbb{Z}} A^M_{(i)} z^{-i-1}$.

The pair $\left(M, Y^M(\cdot, z)\right)$ must satisfy the following axioms:

(a) for all $A \in V$ and $v \in M$, one has $A^M_{(i)}v = 0$, for $i >> 0$;

(b) $Y^M\left(\mathbf{1}^V, z\right) = \mathrm{id}_M$;

(c) for all $A, B \in V$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $v \in M$

$$(z-w)^N \left[Y^M(A,z), Y^M(B,w)\right] v = 0;$$

(d) for all $A \in V$, $v \in M$, there exists $N \in \mathbb{Z}_{\geq 0}$, such that for all $B \in V$,

$$(w+z)^N \left(Y^M(Y(A,w)B,z) - Y^M(A,w+z)Y^M(B,z)\right) v = 0;$$

(e) For $Y^M(\omega, z) = \sum_{i\in\mathbb{Z}} \omega^M_{(i)} z^{-i-1}$, one has

$$\left[\omega^M_{(p+1)}, \omega^M_{(q+1)}\right] = (p-q)\,\omega^M_{(p+q+1)} + \frac{c}{12}\,\delta_{p+q,0}\,(p^3-p)\,\mathrm{id}_M,$$

where $c \in \mathbb{C}$ is the central charge of V. We identify $\omega_{(p+1)} \in \mathrm{End}(M)$ with an action of $L_p \in \mathrm{Vir}$ on M. Moreover $Y^M\left(L_{-1}A, z\right) = \partial_z Y^M(a,z)$.

In the literature, axiom (c) is referred to as *weak commutativity*, and axiom (d) as *weak associativity*. Weak associativity and weak commutativity are known to be equivalent to the *Jacobi identity* (see e.g., [DL, FHL, LL, Li1]).

A weak V-module is *admissible* if in addition one has

(f) $M = \oplus_{i\geq 0} M_i$ is $\mathbb{Z}_{\geq 0}$-graded and $A^M_{(i)} M_k \subseteq M_{k+\deg(A)-i-1}$ for every homogeneous $A \in V$.

We refer to admissible V-modules simply as V-modules. Note that V is a V-module (see [LL, Thm 3.5.4] or [FBZ, §3.2.1]). See §1.5 for an alternative description of V-modules.

1.3. **The Lie algebra ancillary to V.** Given a formal variable t, in this paper we refer to the quotient

$$\mathfrak{L}_t(V) := \left(V \otimes \mathbb{C}(\!(t)\!)\right)/\mathrm{Im}\,\partial,$$

as the *Lie algebra* $\mathfrak{L}(V) = \mathfrak{L}_t(V)$ *ancillary to* V. Here

$$\partial := L_{-1} \otimes \mathrm{id}_{\mathbb{C}(\!(t)\!)} + \mathrm{id}_V \otimes \partial_t. \tag{2}$$

Denote by $A_{[i]}$ the projection in $\mathfrak{L}(V)$ of $A \otimes t^i \in V \otimes \mathbb{C}(\!(t)\!)$. Series of type $\sum_{i\geq i_0} c_i\, A_{[i]}$, for $A \in V$, $c_i \in \mathbb{C}$, and $i_0 \in \mathbb{Z}$, form a spanning set for $\mathfrak{L}(V)$. The Lie bracket of $\mathfrak{L}(V)$ is induced by

$$\left[A_{[i]}, B_{[j]}\right] := \sum_{k\geq 0} \binom{i}{k} \left(A_{(k)} \cdot B\right)_{[i+j-k]}. \tag{3}$$

The axiom on the vacuum vector $\mathbf{1}^V$ implies that $\mathbf{1}^V_{[-1]}$ is central. The Lie algebra $\mathfrak{L}(V)$ is isomorphic to the current Lie algebra in [NT]. In the following, we will use, as formal variable t, a formal coordinate at a point P on a curve, and we will denote $\mathfrak{L}_P(V) = \mathfrak{L}_t(V)$. The Lie algebra $\mathfrak{L}_t(V)$ has a coordinate-free description, discussed in §2.8.

1.4. **The universal enveloping algebra $\mathscr{U}(V)$.** For a vertex algebra V, there is a complete topological associative algebra $\mathscr{U}(V)$, defined originally by Frenkel and Zhu [FZ]. We review it here following the presentation in [FBZ]. Consider the universal enveloping algebra $U(\mathfrak{L}(V))$ of $\mathfrak{L}(V)$, and its completion

$$\widetilde{U}(\mathfrak{L}(V)) := \varprojlim_{N} \, U(\mathfrak{L}(V))/I_N,$$

where I_N is the left ideal generated by $A_{[i]}$, for $A \in V$, and $i > N$. The *universal enveloping algebra* $\mathscr{U}(V)$ of V is defined as the quotient of $\widetilde{U}(\mathfrak{L}(V))$ by the two-sided ideal generated by the Fourier coefficients of the series

$$Y\left[A_{(-1)}B, z\right] \;-\; :Y[A,z]\, Y[B,z]:, \quad \text{for all } A,\ B \in V,$$

where $Y[A,z] = \sum_{i\in\mathbb{Z}} A_{[i]} z^{-i-1}$, and the normal ordering $:\ \ :$ is defined as usual (see e.g., [FLM2]).

For an affine vertex algebra $V = V_\ell(\mathfrak{g})$, one has that $\mathscr{U}(V)$ is isomorphic to a completion $\widetilde{U_\ell}(\widehat{\mathfrak{g}})$ of $U_\ell(\widehat{\mathfrak{g}})$, for all levels $\ell \in \mathbb{C}$. Here, $U_\ell(\widehat{\mathfrak{g}})$ is the quotient of $U(\widehat{\mathfrak{g}})$ by the two-sided ideal generated by $K - \ell$, where $K \in \widehat{\mathfrak{g}}/\mathfrak{g} \otimes \mathbb{C}(\!(t)\!)$, and $\widetilde{U_\ell}(\widehat{\mathfrak{g}})$ is defined as

$$\widetilde{U_\ell}(\widehat{\mathfrak{g}}) := \varprojlim_{N} U_\ell(\widehat{\mathfrak{g}})/U_\ell(\widehat{\mathfrak{g}}) \cdot \mathfrak{g} \otimes t^N \mathbb{C}[\![t]\!]$$

(see e.g., [FBZ, §4.3.2]). The same holds for vertex algebras V which are induced representations of a Lie algebra $\mathfrak{g}$, such as the Heisenberg and Virasoro vertex algebras. In these cases, $\mathscr{U}(V)$ is isomorphic to a completion of $U(\mathfrak{g})$ [FBZ, §5.1.8].

1.5. **Action on V-modules.** Both $\mathfrak{L}(V)$ and $\mathscr{U}(V)$ act on any V-module M via the Lie algebra homomorphism $\mathfrak{L}(V) \to \mathrm{End}(M)$ obtained by mapping $A_{[i]}$ to the Fourier coefficient $A_{(i)}$ of the vertex operator $Y^M(A,z) = \sum_i A_{(i)} z^{-i-1}$. More generally, the series $\sum_{i\geq i_0} c_i\, A_{[i]}$ acts on a V-module M via

$$\mathrm{Res}_{z=0}\, Y^M(A,z) \sum_{i\geq i_0} c_i z^i dz.$$

We note that an $\mathfrak{L}(V)$-module need not be a V-module. On the other hand, there is an equivalence between the categories of weak V-modules and smooth $\mathscr{U}(V)$-modules (see [FBZ, §5.1.6]). We recall that an $\mathscr{U}(V)$-module M is *smooth* if for any $w \in M$ and $A \in V$, one has $A_{[i]}w = 0$ for $i >> 0$.

In [NT], Nagatomo and Tsuchiya consider $\mathscr{U}(V)$-modules M such that: (i) M is finitely generated over $\mathscr{U}(V)$; (ii) $F^0(\mathscr{U}(V))v$ is finite-dimensional, for every $v \in M$; and (iii) for every $v \in M$, there exists a positive integer k such that $F^k(\mathscr{U}(V))v = 0$. Here $F^k(\mathscr{U}(V)) \subset \mathscr{U}(V)$ is the vector subspace topologically generated by compositions of operators with total degree $\leq k$. These conditions ensure that the modules in [NT] are admissible V-modules. In the following, a V-module is finitely generated if it is finitely generated as an $\mathscr{U}(V)$-module.

Below we review how to describe irreducible V-modules.

1.6. **Correspondence between V-modules and $A(V)$-modules.** A V-module W is *irreducible*, or *simple*, if it has no sub-representation other than the trivial representation 0 and W itself. We review here Zhu's associative algebra $A(V)$, and the one-to-one correspondence between isomorphism classes of finite-dimensional irreducible $A(V)$-modules and isomorphism classes of irreducible V-modules [Zhu2].

Zhu's algebra is the quotient $A(V) := V/O(V)$, where $O(V)$ is the vector subspace of V linearly spanned by elements of the form

$$\mathrm{Res}_{z=0} \frac{(1+z)^{\deg A}}{z^2} Y(A,z)B,$$

where A is homogeneous in V. The image of an element $A \in V$ in $A(V)$ is denoted by $o(A)$. The product in $A(V)$ is defined by

$$o(A) * o(B) = \mathrm{Res}_{z=0} \frac{(1+z)^{\deg A}}{z} Y(A,z)B,$$

for homogeneous A in V. Nagatomo and Tsuchiya [NT] consider an isomorphic copy of $A(V)$, which they refer to as the *zero-mode algebra*.

Given a V-module $W = \oplus_{i\geq 0} W_i$, one has that W_0 is an $A(V)$-module [Zhu2, Thm 2.2.2]. The action of $A(V)$ on W_0 is defined as follows: an element $o(A) \in A(V)$, image of a homogeneous element $A \in V$, acts on W_0 as the endomorphism $A_{(\deg A-1)}$, a Fourier coefficient of $Y^W(A,z)$.

Conversely, one constructs a V-module from an $A(V)$-module in the following way. The Lie algebra $\mathfrak{L}(V)$ admits a triangular decomposition:

$$\mathfrak{L}(V) = \mathfrak{L}(V)_{<0} \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{>0}, \tag{4}$$

determined by the degree (1), that is, $\deg(A_{[i]}) := \deg(A) - i - 1$, for homogeneous $A \in V$. From the definition of the Lie bracket (3) of $\mathfrak{L}(V)$, one checks that each summand above is a Lie subalgebra of $\mathfrak{L}(V)$. This induces a subalgebra $\mathscr{U}(V)_{\leq 0}$ of $\mathscr{U}(V)$.

Given a finite-dimensional $A(V)$-module E, the *generalized Verma $\mathscr{U}(V)$-module* is

$$M(E) := \mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E.$$

To make E into an $\mathscr{U}(V)_{\leq 0}$-module, one lets $\mathfrak{L}(V)_{<0}$ act trivially on E, and $\mathfrak{L}(V)_0$ act by the homomorphism of Lie algebras $\mathfrak{L}(V)_0 \twoheadrightarrow A(V)_{\mathrm{Lie}}$ induced by the identity endomorphism of V [Li2, Lemma 3.2.1]. For homogeneous $A \in V_k$, the image of the element $A_{[k-1]} \in \mathfrak{L}(V)_0$ in $A(V)$ is $o(A)$. By construction, $M(E)$ is automatically a V-module.

Given an *irreducible* V-module $W = \oplus_{i\geq 0} W_i$, the space W_0 is a finite-dimensional *irreducible* $A(V)$-module; conversely, given a finite-dimensional *irreducible* $A(V)$-module E, there is a unique maximal proper V-submodule $N(E)$ of the V-module $M(E)$ with $N(E) \cap E = 0$ such that $L(E) = M(E)/N(E)$ is an *irreducible* V-module [Zhu2].

1.7. **Rational vertex algebras.** A vertex algebra V is *rational* if every finitely generated V-module is a direct sum of irreducible V-modules. A rational vertex algebra has only finitely many isomorphism classes of irreducible modules, and an irreducible module $M = \oplus_{i\geq 0} M_i$ over a rational vertex algebra satisfies $\dim M_i < \infty$ [DLM3].

An *ordinary* V-module is a weak V-module which carries a $\mathbb{C}$-grading $M = \oplus_{\lambda\in\mathbb{C}} M_\lambda$ such that: $L_0|_{M_\lambda} = \lambda\, \mathrm{id}_{M_\lambda}$; $\dim M_\lambda < \infty$; and for fixed λ, one has $M_{\lambda+n} = 0$, for integers $n << 0$. Ordinary V-modules are admissible, and when V is rational, every simple admissible V-module is ordinary [DLM3], [DLM2, Rmk 2.4]. It follows that for rational V, finitely generated V-modules are direct sum of simple ordinary V-modules. In particular, a finitely generated admissible module $M = \oplus_{i\geq 0} M_i$ over a rational vertex algebra satisfies $\dim M_i < \infty$, for all i, and L_0 acts semisimply on such M.

When V is rational, the associative algebra $A(V)$ is semisimple [Zhu2]. For a rational vertex algebra V, given a simple module E over Zhu's algebra $A(V)$, the Verma module $M(E)$ remains simple. In general, Verma modules are not necessarily simple, but they are indecomposable. Hence complete reducibility implies that simple and indecomposable coincide.

Remark 1.7.1. Nagatomo and Tsuchiya asked whether an irreducible $A(V)$-module E always gives rise to an *irreducible* V-module $M(E) = L(E)$ when V is C_2-cofinite and $A(V)$ is semisimple [NT, Problem, page 439]. They pointed out that this is indeed the case for rational Virasoro vertex algebras and for irreducible affine vertex algebras $L_\ell(\mathfrak{g})$ defined by simple Lie algebras $\mathfrak{g}$ at level ℓ (see e.g., [NT, §A.1] and §9). As remarked above, given a rational vertex operator algebra, this irreducibility condition always holds for generalized Verma modules, which are indecomposable, since for a rational vertex algebra, simple and indecomposable modules coincide.

1.8. **Contragredient modules.** Essential to the Factorization Theorem, contragredient modules are reviewed here following [FHL, §5.2].

Let V be a vertex algebra. Given a V-module $\left(M = \oplus_{i\geq 0} M_i, Y^M(-,z)\right)$, its *contragredient module* is $\left(M', Y^{M'}(-,z)\right)$, where M' is the graded dual of M, that is, $M' := \oplus_{i\geq 0} M_i^\vee$, with $M_i^\vee := \mathrm{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, and

$$Y^{M'}(-,z)\colon V \to \mathrm{End}\left(M'\right) [\![z, z^{-1}]\!]$$

is the unique linear map determined by

$$\left\langle Y^{M'}(A,z)\psi, m\right\rangle = \left\langle \psi, Y^M\left(e^{zL_1}(-z^{-2})^{L_0} A, z^{-1}\right) m\right\rangle \tag{5}$$

for $A \in V$, $\psi \in M'$, and $m \in M$. Here and throughout, $\langle\cdot,\cdot\rangle$ is the natural pairing between a vector space and its graded dual.

1.8.1. In view of the correspondence between isomorphism classes of irreducible V-modules and irreducible $A(V)$-modules (§1.5), we explicitly describe the $A(V)$-module structure on $M_0^\vee$, with M a V-module. For this purpose, consider the involution $\vartheta\colon \mathfrak{L}(V) \to \mathfrak{L}(V)$ induced from

$$\vartheta\left(A_{[j]}\right) := (-1)^{k-1} \sum_{i \geq 0} \frac{1}{i!} L_1^i A_{[2k-j-i-2]} \tag{6}$$

for a homogeneous element $A \in V_k$. Observe that since the operator L_1 has negative degree, the above sum is finite. This involution appeared in [Bor], and it naturally arises from the action of the vertex operators on the contragredient module V'. Since ϑ restricts to an involution of $\mathfrak{L}(V)_0$ leaving $O(V)$ invariant, it induces an involution on Zhu's algebra $A(V)$. The following statement is a direct consequence of the definition of contragredient modules.

Lemma 1.8.1. *i) The image of $\psi \in M_0^\vee$ under the action of $\sigma \in \mathfrak{L}(V)$ is the linear functional*

$$\sigma \cdot \psi = -\psi \circ \vartheta(\sigma).$$

ii) The image of $\psi \in M_0^\vee$ under the action of $o(A) \in A(V)$ is

$$o(A) \cdot \psi = -\psi \circ \vartheta(o(A)).$$

For homogeneous $A \in V$ of degree k, and for $m \in M_0$, this is

$$\langle o(A) \cdot \psi, m \rangle = (-1)^k \left\langle \psi, \sum_{i \geq 0} \frac{1}{i!} \left(L_1^i A\right)_{(k-i-1)} m \right\rangle.$$

We refer the reader to [FBZ, §10.4.8] for a geometric realization of the involution ϑ. In §3, the involution ϑ will be needed to describe chiral Lie algebras on nodal curves.

1.9. **Stable k-differentials.** Let $(C, P_\bullet)$ be a stable n-pointed curve with at least one node, and let ω_C be the dualizing sheaf on C. We review here *(stable) k-differentials* on C, that is, sections of $\omega_C^{\otimes k}$, for an integer k. When $k \geq 1$, by $\omega_C^{\otimes -k}$ we mean $(\omega_C^\vee)^{\otimes k}$.

Let $\widetilde{C} \to C$ be the partial normalization of C at a node Q, let $Q_+, Q_- \in \widetilde{C}$ be the two preimages of Q, and set $Q_\bullet = (Q_+, Q_-)$. Note that the curve $\widetilde{C}$ may not be connected. Let s_+ and s_- be formal coordinates at the points Q_+ and Q_-, respectively. We write $Q_\pm$ to denote either point, and similarly use $s_\pm$ to denote either formal coordinate. For a section

$$\mu \in H^0\left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \omega_{\widetilde{C}}^{\otimes k}\right) =: \widetilde{\Gamma},$$

let $\mu_{Q_\pm} \in \mathbb{C}(\!(s_\pm)\!)(ds_\pm)^k$ be the Laurent series expansion of μ at $Q_\pm$, that is, the image of μ under the restriction morphism

$$H^0\left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \omega_{\widetilde{C}}^{\otimes k}\right) \to H^0\left(D_{Q_\pm}^\times, \omega_{\widetilde{C}}^{\otimes k}\right) \simeq_{s_\pm} \mathbb{C}(\!(s_\pm)\!)(ds_\pm)^k.$$

Here $D_{Q_\pm}^\times$ is the punctured formal disk about $Q_\pm$, that is, the spectrum of the field of fractions of the completed local ring $\widehat{\mathscr{O}}_{Q_\pm}$, and $\simeq_{s_\pm}$ denotes the isomorphism given by fixing the formal coordinate $s_\pm$ at $Q_\pm$.

For a k-differential μ, define the *order* $\mathrm{ord}_{Q_\pm}(\mu)$ of μ at the point $Q_\pm$ as the highest integer m such that $\mu_{Q_\pm} \in s_\pm^m \mathbb{C}[\![s_\pm]\!](ds_\pm)^k$. For a k-differential μ with $\mathrm{ord}_{Q_\pm}(\mu) \geq -k$, the *$k$-residue* $\mathrm{Res}^k_{Q_\pm}(\mu)$ of μ at the point $Q_\pm$ is defined as the coefficient of $s_\pm^{-k}(ds_\pm)^k$ in $\mu_{Q_\pm}$.

The order and the k-residue at $Q_\pm$ are independent of the formal coordinate $s_\pm$ at $Q_\pm$. For the definition of the k-residue without the assumption $\mathrm{ord}_{Q_\pm}(\mu) \geq -k$, see e.g., [BCG$^+$]; here we only need the case $\mathrm{ord}_{Q_\pm}(\mu) \geq -k$.

Lemma 1.9.1. *Assume that $C \setminus P_\bullet$ is affine. For all integers k, one has*

$$H^0\left(C \setminus P_\bullet, \omega_C^{\otimes k}\right) = \left\{ \mu \in \widetilde{\Gamma} \;\middle|\; \begin{array}{l} \mathrm{ord}_{Q_+}(\mu) \geq -k,\, \mathrm{ord}_{Q_-}(\mu) \geq -k, \\ \mathrm{Res}^k_{Q_+}(\mu) = (-1)^k \,\mathrm{Res}^k_{Q_-}(\mu) \end{array} \right\}.$$

Proof. It is enough to prove the statement for $k \in \{-1, 0, 1\}$: indeed, for negative integers k, sections of $\omega_C^{\otimes k}$ on the affine $C \setminus P_\bullet$ are tensor products of sections of ω_C^{-1}, and the Laurent series expansions of sections of $\omega_{\widetilde{C}}^{\otimes k}$ at Q_+ and Q_- are obtained as tensors of the Laurent series expansions of sections of $\omega_{\widetilde{C}}^{-1}$ at Q_+ and Q_-, respectively. Similarly, for positive integers k.

When $k = 1$, the statement is about sections of ω_C, and by definition sections of ω_C are sections of $\omega_{\widetilde{C}}$ with at most simple poles at Q_+ and Q_- such that $\mathrm{Res}_{Q_+}(\mu) + \mathrm{Res}_{Q_-}(\mu) = 0$. When $k = 0$, the statement is about sections of $\mathscr{O}_C$, and indeed a regular function on C is a regular function μ on $\widetilde{C}$ such that $\mu(Q_+) = \mu(Q_-)$. When $k = -1$, by definition we have $\omega_C^{-1} = \mathscr{H}\mathrm{om}_{\mathscr{O}_C}(\omega_C, \mathscr{O}_C)$, and the statement follows from a direct computation using the cases $k \in \{0, 1\}$. □

1.10. **A consequence of Riemann-Roch.** Throughout this paper, we will use a consequence of the Riemann-Roch theorem which we next describe.

Let C be a smooth curve, possibly disconnected, with two non-empty sets of distinct marked points $P_\bullet = (P_1, \ldots, P_n)$ and $Q_\bullet = (Q_1, \ldots, Q_m)$. Assume that each irreducible component of C contains at least one of the marked points $P_\bullet$. In particular, this will imply that $C \setminus P_\bullet$ is affine. Let s_i be a formal coordinate at the point Q_i, for each i. Fix an integer k. For all integers d and N, there exists $\mu \in H^0\left(C \setminus P_\bullet \sqcup Q_\bullet, \omega_C^{\otimes k}\right)$ such that its Laurent series expansions at the points $Q_\bullet$ satisfy:

$$\begin{aligned} \mu_{Q_i} &\equiv s_i^d (ds_i)^k && \in \quad \mathbb{C}(\!(s_i)\!)(ds_i)^k / s_i^N \mathbb{C}[\![s_i]\!](ds_i)^k, && \text{for a fixed } i, \\ \mu_{Q_j} &\equiv 0 && \in \quad \mathbb{C}(\!(s_j)\!)(ds_j)^k / s_j^N \mathbb{C}[\![s_j]\!](ds_j)^k, && \text{for all } j \neq i. \end{aligned}$$

This statement has appeared for instance in [Zhu1].

2. Sheaves of vertex algebras on stable curves

Here we shall describe the new sheaf of vertex algebras $\mathcal{V}_C$ on a stable curve C. The sheaf $\mathcal{V}_C$ (§2.5) and its flat connection (§2.7) allow one to give a coordinate-free description of the Lie algebra ancillary to V (§2.8) and to construct the chiral Lie algebra (§3), whose representations give rise to vector spaces of coinvariants (§4). The sheaf $\mathcal{V}_C$ for smooth C was introduced in [FBZ]. The extension to nodal curves was first given in [DGT1] using a different sheaf $\mathscr{V}_C$. The results obtained in [DGT1] for $\mathscr{V}_C$ are recovered here using $\mathcal{V}_C$ in place of $\mathscr{V}_C$.

2.1. **The group scheme $\operatorname{Aut}\mathcal{O}$.** Consider the functor which assigns to a $\mathbb{C}$-algebra R the Lie group:

$$\operatorname{Aut}\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = a_1 z + a_2 z^2 + \cdots \mid a_i \in R,\, a_1 \text{ a unit} \right\}$$

of continuous automorphisms of the algebra $R[\![z]\!]$ preserving the ideal $zR[\![z]\!]$. The group law is given by composition of series: $\rho_1 \cdot \rho_2 := \rho_2 \circ \rho_1$. This functor is represented by a group scheme, denoted $\operatorname{Aut}\mathcal{O}$.

To construct the sheaf of vertex algebras $\mathcal{V}_C$ on a stable curve C, we describe below the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\mathscr{A}ut_C \to C$, and actions of $\operatorname{Aut}\mathcal{O}$ on the vertex operator algebra V and on $\mathscr{A}ut_C \times V$.

2.2. **Coordinatized curves.** Assume first that C is a *smooth* curve. Let $\mathscr{A}ut_C$ be the infinite-dimensional smooth variety whose points consist of pairs (P, t), where P is a point in C, and t is a formal coordinate at P (see [ADCKP]). A formal coordinate t at P is an element of the completed local ring $\widehat{\mathscr{O}}_P$ such that $t \in \mathfrak{m}_P/\mathfrak{m}_P^2$, where $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathscr{O}}_P$. There is a natural forgetful map $\mathscr{A}ut_C \to C$, with fiber at a point $P \in C$ equal to the set of formal coordinates at P:

$$\mathscr{A}ut_P = \left\{ t \in \widehat{\mathscr{O}}_P \mid t \in \mathfrak{m}_P/\mathfrak{m}_P^2 \right\}.$$

The group scheme $\operatorname{Aut}\mathcal{O}$ has a right action on the fibers of $\mathscr{A}ut_C \to C$ by change of coordinates:

$$\mathscr{A}ut_C \times \operatorname{Aut}\mathcal{O} \to \mathscr{A}ut_C, \qquad ((P,t),\rho) \mapsto (P, t\cdot\rho := \rho(t)).$$

This action is simply transitive, thus $\mathscr{A}ut_C$ is a principal $(\operatorname{Aut}\mathcal{O})$-bundle on C. The choice of a formal coordinate t at P gives rise to the trivialization

$$\operatorname{Aut}\mathcal{O} \xrightarrow{\simeq_t} \mathscr{A}ut_P, \qquad \rho \mapsto \rho(t).$$

For a *nodal* curve C, let $\widetilde{C} \to C$ denote the normalization of C. Assume for simplicity that C has only one node Q, and let Q_+ and Q_- be the two preimages in $\widetilde{C}$ of Q. A choice of formal coordinates s_+ and s_- at Q_+ and Q_-, respectively, determines a smoothing of the nodal curve C over $\operatorname{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+ s_- = q$ (see §8.1 for more details). A principal $(\operatorname{Aut}\mathcal{O})$-bundle on a nodal curve is equivalent to the datum of a principal $(\operatorname{Aut}\mathcal{O})$-bundle on its normalization together with a gluing isomorphism between the fibers over the two preimages of each node. In particular, one constructs the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\mathscr{A}ut_C$ on C from the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\mathscr{A}ut_{\widetilde{C}}$ on $\widetilde{C}$

by identifying the fibers at the two preimages Q_+ and Q_- of the node Q in C by the following gluing isomorphism:

$$\mathscr{A}ut_{Q_+} \simeq_{s_+} \operatorname{Aut}\mathcal{O} \xrightarrow{\cong} \operatorname{Aut}\mathcal{O} \simeq_{s_-} \mathscr{A}ut_{Q_-}, \qquad \rho(s_+) \mapsto \rho \circ \gamma(s_-), \tag{7}$$

where $\gamma \in \operatorname{Aut}\mathcal{O}$ is defined as

$$\gamma(z) := \frac{1}{1+z} - 1 = -z + z^2 - z^3 + \cdots . \tag{8}$$

Note that $(\gamma \circ \gamma)(z) = z$, hence (7) determines an involution of $\operatorname{Aut}\mathcal{O}$. The isomorphism (7) is induced from the identification $s_+ = \gamma(s_-)$. In §2.2.1 we explain how this construction extends in families.

2.2.1. The above construction of $\mathscr{A}ut_C$ can be carried out in families of stable curves. Let $\overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal curve over the moduli space of stable curves of genus g. Extending the construction of $\mathscr{A}ut_C$ over $\overline{\mathcal{M}}_g$, one obtains a principal $(\operatorname{Aut}\mathcal{O})$-bundle $\widehat{\overline{\mathcal{C}}}_g$ over $\overline{\mathcal{C}}_g$:

$$\begin{array}{ccc}
\mathscr{A}ut_C & \longrightarrow & \widehat{\overline{\mathcal{C}}}_g \\
\downarrow & & \downarrow{\scriptstyle \operatorname{Aut}\mathcal{O}} \\
C & \longrightarrow & \overline{\mathcal{C}}_g \\
\downarrow & & \downarrow \\
\operatorname{Spec}(\mathbb{C}) & \xrightarrow{[C]} & \overline{\mathcal{M}}_g .
\end{array}$$

For a stable curve C of genus g, $\mathscr{A}ut_C$ is the fiber of the projection $\widehat{\overline{\mathcal{C}}}_g \to \overline{\mathcal{M}}_g$ over the point $[C] \in \overline{\mathcal{M}}_g$.

It is sometimes convenient to identify the universal curve $\overline{\mathcal{C}}_g$ with the moduli space of stable pointed curves $\overline{\mathcal{M}}_{g,1}$. This identifications induces an identification of the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\widehat{\overline{\mathcal{C}}}_g \to \overline{\mathcal{C}}_g$ with a principal $(\operatorname{Aut}\mathcal{O})$-bundle $\widehat{\overline{\mathcal{M}}}_{g,1} \to \overline{\mathcal{M}}_{g,1}$. Here $\widehat{\overline{\mathcal{M}}}_{g,1}$ is the moduli space of objects of type (C, P, t), where (C, P) is a stable pointed curve of genus g, and t is a formal coordinate at P. For smooth curves, this moduli space is constructed in [ADCKP]. One has the following diagram:

$$\begin{array}{ccccc}
\mathscr{A}ut_P & \longrightarrow & \mathscr{A}ut_C & \longrightarrow & \widehat{\overline{\mathcal{M}}}_{g,1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \operatorname{Aut}\mathcal{O}} \\
\operatorname{Spec}(\mathbb{C}) & \xrightarrow{[C,P]} & C & \longrightarrow & \overline{\mathcal{M}}_{g,1} \\
 & & \downarrow & & \downarrow \\
 & & \operatorname{Spec}(\mathbb{C}) & \xrightarrow{[C]} & \overline{\mathcal{M}}_g .
\end{array}$$

Let C be a stable curve, and assume for simplicity that C has only one node Q. Over the singular point $Q \in C$, the fiber $\mathscr{A}ut_Q$ can be identified with the space of formal coordinates at the point Q in C', where (C', Q) is formed by stable reduction at the unstable pointed curve (C, Q). Stable reduction is carried out by blowing-up. As a result, C' consists of the partial normalization $\widetilde{C}$ of C at Q together with a rational exceptional component meeting $\widetilde{C}$ transversally at the two preimages Q_+ and $Q_- \in \widetilde{C}$ of the node Q. Such a rational component contains three special points: the two attaching points and a point labelled Q. Up to isomorphism, one can identify this rational component with $\mathbb{P}^1$, with the points attached to Q_+ and Q_- identified with 0 and $\infty \in \mathbb{P}^1$, respectively, and the point Q identified with $1 \in \mathbb{P}^1$. The fiber $\mathscr{A}ut_Q$ of $\mathscr{A}ut_C$ over the node Q in C is identified with the space of formal coordinates at the point 1 in $\mathbb{P}^1 \subset C'$.

Here are more details. Choose formal coordinates s_+ and s_- at Q_+ and Q_-, respectively. As mentioned in the previous section, a choice of such coordinates determines a smoothing of C over $\mathrm{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+s_- = q$. After blowing-up such a family at the point Q, locally around the two resulting nodes incident to the exceptional component $\mathbb{P}^1$, the curve C' at $q = 0$ is given by equations $s_+s_0 = 0$ and $s_\infty s_- = 0$, where s_0, s_∞ are formal coordinates at $0, \infty \in \mathbb{P}^1$, respectively. The formal coordinates s_0 and s_∞ satisfy $s_0 s_\infty = 1$. A pair of such coordinates s_0, s_∞ induce formal coordinates $s_0 - 1$ and $s_\infty - 1$ at $1 \in \mathbb{P}^1$, with change of coordinates given precisely by $s_0 - 1 = \gamma(s_\infty - 1)$, where γ is as in (8). It follows that for the nodal curve C', one has the identifications $s_+ = \gamma(s_0)$, $s_- = \gamma(s_\infty)$, and $s_0 - 1 = \gamma(s_\infty - 1)$. These identifications determine the identification of the space $\mathscr{A}ut_C$ with the fiber of the projection $\widehat{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ over the point $[C] \in \overline{\mathcal{M}}_g$.

For further information about $\widehat{\mathcal{M}}_{g,1}$, see [FBZ, §6.5] for the case of smooth curves. We use the construction here described of $\widehat{\mathcal{M}}_{g,1}$ in [DGT1, §2].

2.3. **Action of $\mathrm{Aut}\,\mathcal{O}$ on V.** The functor which assigns to a $\mathbb{C}$-algebra R the Lie algebra of the Lie group $\mathrm{Aut}\,\mathcal{O}(R)$ is

$$\mathrm{Der}_0\,\mathcal{O}(R) = \mathrm{Lie}(\mathrm{Aut}\,\mathcal{O})(R) = R[\![z]\!]z\partial_z.$$

The Lie algebra $\mathrm{Der}_0\,\mathcal{O}(R)$ is generated over R by the Virasoro elements L_p, for $p \geq 0$, thus $\mathrm{Der}_0\,\mathcal{O}$ is a Lie subalgebra of the Virasoro Lie algebra.

The action of the Virasoro Lie algebra on a vertex operator algebra V restricts to an action of Der_0 on V. One can integrate this action to get a left action of $\mathrm{Aut}\,\mathcal{O}$ on V defined as the inductive limit of the actions on the subspaces $V_{\leq k} := \oplus_{i \leq k} V_i$. This follows from the fact that L_0 acts semisimply with integral eigenvalues, and L_p acts locally nilpotently for $p > 0$ [FBZ, §6.3].

Explicitly, to compute the action on V of an an element $\rho \in \operatorname{Aut}\mathcal{O}$, one proceeds as follows. The element $\rho(z)$ can be expressed as

$$\rho(z) = \exp\left(\sum_{i\geq 0} a_i z^{i+1}\partial_z\right)(z)$$

for some $a_i \in \mathbb{C}$ (see e.g., [FBZ, §6.3.1]). Assuming $0 \leq \operatorname{Im}(a_0) < 2\pi$, the coefficients a_i are uniquely determined. Hence, ρ acts on V as $\exp\left(\sum_{i\geq 0} -a_i L_i\right)$.

As an example and for later use, the element $\gamma \in \operatorname{Aut}\mathcal{O}$ from (8) can be expressed as

$$\gamma(z) = e^{-z^2\partial_z}(-1)^{-z\partial_z}(z). \tag{9}$$

This is a special case of the computation in [FBZ, (10.4.3)]. Thus γ acts on V as $e^{L_1}(-1)^{L_0}$. This element allows one to express the gluing isomorphism for the vertex algebra bundle below.

2.4. **Action of $\operatorname{Aut}\mathcal{O}$ on $\mathscr{A}ut_C \times V$.** The group $\operatorname{Aut}\mathcal{O}$ has a right equivariant action on the trivial bundle $\mathscr{A}ut_C \times V \to \mathscr{A}ut_C$ defined by

$$(P, t, A) \cdot \rho = \left(P, \rho(t), \rho^{-1} \cdot A\right),$$

for $\rho \in \operatorname{Aut}\mathcal{O}$ and $(P, t, A) \in \mathscr{A}ut_C \times V$.

2.5. **The sheaf of vertex algebras.** Assume first that the curve C is *smooth*. The quotient of $\mathscr{A}ut_C \times V$ by the action of $\operatorname{Aut}\mathcal{O}$ descends along the map $\mathscr{A}ut_C \to C$ to the *vertex algebra bundle* V_C on C:

$$\begin{array}{ccc} \mathscr{A}ut_C \times V & \longrightarrow & \mathscr{A}ut_C \times_{\operatorname{Aut}\mathcal{O}} V =: V_C \\ \downarrow & & \downarrow \\ \mathscr{A}ut_C & \xrightarrow[\pi]{\operatorname{Aut}\mathcal{O}} & C. \end{array}$$

In V_C, one has identities

$$(P, t, A) = \left(P, \rho(t), \rho^{-1} \cdot A\right), \tag{10}$$

for $\rho \in \operatorname{Aut}\mathcal{O}$ and $(P, t, A) \in \mathscr{A}ut_C \times V$. The sheaf of sections of V_C is the sheaf of vertex algebras:

$$\mathcal{V}_C := \left(V \otimes \pi_* \mathscr{O}_{\mathscr{A}ut_C}\right)^{\operatorname{Aut}\mathcal{O}}.$$

This is a locally free sheaf of $\mathscr{O}_C$-modules on C. The stalk of $\mathcal{V}_C$ at a point $P \in C$ is isomorphic to $\mathscr{A}ut_P \times_{\operatorname{Aut}\mathcal{O}} V$. Given a formal coordinate t at P, one has the trivialization

$$\mathscr{A}ut_P \times_{\operatorname{Aut}\mathcal{O}} V \simeq_t V.$$

For a *nodal* curve C, the sheaf $\mathcal{V}_C$ is defined as follows. Assume for simplicity that C has only one node Q. Let $\nu\colon \widetilde{C} \to C$ be the normalization of C, and let Q_+ and Q_- in $\widetilde{C}$ be the two preimages of Q. Choose formal coordinates s_+ and s_- at Q_+ and Q_-, respectively. Consider the sheaf

$$\widetilde{\mathcal{V}} := \left(\oplus_{k\geq 0} V_k \otimes \mathscr{O}_{\widetilde{C}}(-kQ_+ - kQ_-) \otimes \pi_* \mathscr{O}_{\mathscr{A}ut_{\widetilde{C}}}\right)^{\operatorname{Aut}\mathcal{O}}. \tag{11}$$

We define $\mathcal{V}_C$ as the subsheaf of $\nu_*(\widetilde{\mathcal{V}})$ obtained by identifying the stalks at Q_+ and Q_- of $\widetilde{\mathcal{V}}$ via the gluing isomorphism:

$$\begin{array}{ccc} V & \dashrightarrow & V \\ {\scriptstyle \simeq_{s_+}}\downarrow & & \uparrow{\scriptstyle \simeq_{s_-}} \\ \mathscr{A}ut_{Q_+} \times_{\operatorname{Aut}\mathcal{O}} V & \xrightarrow{\cong} & \mathscr{A}ut_{Q_-} \times_{\operatorname{Aut}\mathcal{O}} V \end{array}$$

given by

$$A \mapsto e^{L_1}(-1)^{L_0} A.$$

The operator $e^{L_1}(-1)^{L_0}$ defines an involution on V, and coincides with the action on V of the element $\gamma \in \operatorname{Aut}\mathcal{O}$ from (8) and (9). For homogeneous $A \in V$ of degree k, one has

$$\gamma \cdot A = e^{L_1}(-1)^{L_0} A = (-1)^k \sum_{i\geq 0} \frac{1}{i!} L_1^i A.$$

The isomorphism giving gluing for $\mathcal{V}_C$ is induced from the gluing isomorphism for $\mathscr{A}ut_C$ in (7). Indeed, these maps correspond to the identities

$$(Q_+, s_+, A) = (Q_-, \gamma(s_-), A) = (Q_-, s_-, \gamma \cdot A).$$

The equivalence $s_+ = \gamma(s_-)$ follows from (7), and the second equality comes from (10) and the identity $\gamma = \gamma^{-1}$.

Remark 2.5.1. We emphasize that the sheaf $\mathcal{V}_C$ defined here over a nodal curve C does not coincide in general with the sheaf $\mathscr{V}_C$ defined in [DGT1], although they agree when restricted to the complements of the nodes of C. Moreover, the sheaf $\mathcal{V}_C$ is not quasi-coherent. Indeed, combining Lemmas 2.6.1 and 2.6.2, one sees that in order to be quasi-coherent, the gluing of $\mathcal{V}_C$ over nodes should be given by $(-1)^{L_0}$, instead of $\gamma = e^{L_1}(-1)^{L_0}$. The gluing γ allows us to deduce Proposition 3.3.1, a key ingredient in the proof of the factorization theorem.

2.6. **The structure of the sheaf $\mathcal{V}_C$.** We describe here some properties of $\mathcal{V}_C$. For a smooth curve C, the sheaf $\mathcal{V}_C$ is filtered by the sheaves $\mathcal{V}_{\leq k}$ defined as the sheaves of sections of the vector bundles of finite rank $\mathscr{A}ut_C \times_{\operatorname{Aut}\mathcal{O}} V_{\leq k}$. While the action of $\operatorname{Aut}\mathcal{O}$ on $V_{\leq k}$ is well-defined, the action of $\operatorname{Aut}\mathcal{O}$ on V_k is so only modulo $V_{\leq k-1}$, for each k. In particular, $\mathcal{V}_{\leq k}$ is well-defined, but $\mathscr{A}ut_C \times_{\operatorname{Aut}\mathcal{O}} V_k$ only makes sense as a quotient of $\mathscr{A}ut_C \times_{\operatorname{Aut}\mathcal{O}} V_{\leq k}$ modulo $\mathscr{A}ut_C \times_{\operatorname{Aut}\mathcal{O}} V_{\leq k-1}$.

For a nodal curve C, assuming for simplicity that C has only one node Q, the sheaves

$$\left(\oplus_{i=0}^k V_i \otimes \mathscr{O}_{\widetilde{C}}(-i\,Q_+ - i\,Q_-) \otimes \pi_* \mathscr{O}_{\mathscr{A}ut_{\widetilde{C}}}\right)^{\operatorname{Aut}\mathcal{O}}$$

provide a stratification of the sheaf $\widetilde{\mathcal{V}}$ on the normalization $\widetilde{C}$ from (11). The restriction of the gluing isomorphism in §2.5 gives a gluing isomorphism between the stalks of such sheaves at the two preimages Q_+ and Q_- of the node Q, hence such a stratification induces a stratification of $\mathcal{V}_C$. We denote the subsheaves of such a stratification as $\mathcal{V}_{\leq k}$, as in the smooth case.

Consider the associated graded sheaf

$$\mathrm{gr}_\bullet \mathcal{V}_C := \oplus_{k\geq 0} \mathrm{gr}_k \mathcal{V}_C, \qquad \text{where} \qquad \mathrm{gr}_k \mathcal{V}_C := \mathcal{V}_{\leq k}/\mathcal{V}_{\leq k-1}.$$

Lemma 2.6.1. *One has*

$$\mathrm{gr}_\bullet \mathcal{V}_C \cong \oplus_{k\geq 0} \left(\omega_C^{\otimes -k}\right)^{\oplus \dim V_k}.$$

This was proved in [FBZ, §6.5.9] for smooth curves. The argument made there extends to stable curves if one replaces the sheaf of differentials Ω_C^1 with the dualizing sheaf ω_C. We sketch the proof for the reader's convenience.

Proof. Consider V_k as the quotient $(\mathrm{Aut}\,\mathcal{O})$-representation $V_{\leq k}/V_{\leq k-1}$, and let $A \in V_k$ be nonzero. One has $L_0 \cdot A = kA$ and $L_p \cdot A = 0$ in $V_k = V_{\leq k}/V_{\leq k-1}$ for $p > 0$. Assume first that C is smooth. Then it follows that

$$\mathscr{A} := \left(\mathbb{C}A \otimes \pi_* \mathscr{O}_{\mathscr{A}ut_{\widetilde{C}}}\right)^{\mathrm{Aut}\,\mathcal{O}}$$

is a line sub-bundle of $\mathrm{gr}_k \mathcal{V}_C$. From [FBZ, §6.5.9], the transition functions of $\mathscr{A}$ and $\omega_C^{\otimes -k}$ match, hence one concludes that $\mathscr{A} \cong \omega_C^{\otimes -k}$, and this implies the statement.

Next, we consider the case when C is nodal. Assume for simplicity that C has only one node Q. Consider the line bundle $\mathscr{A}$ constructed from the line bundle

$$\mathscr{A}' := \left(\mathbb{C}A \otimes \pi_* \mathscr{O}_{\mathscr{A}ut_{\widetilde{C}}}\right)^{\mathrm{Aut}\,\mathcal{O}} \otimes \mathscr{O}_{\widetilde{C}}(-kQ_+ - kQ_-)$$

on the normalization $\widetilde{C}$ of C by identifying the stalks at the preimages Q_+ and Q_- of the node. From the discussion of the smooth case, one has

$$\mathscr{A}' \cong \omega_{\widetilde{C}}^{\otimes -k}\left(-kQ_+ - kQ_-\right).$$

It remains to determine the isomorphisms used to identify the stalks at Q_+ and Q_-. The gluing isomorphism $A \mapsto e^{L_1}(-1)^{L_0}A$ of $\mathcal{V}_{\leq k}$ from §2.5 induces the gluing $A \mapsto (-1)^{L_0}A$ of $\mathrm{gr}_k \mathcal{V}_C$. In particular, this is the gluing isomorphism of $\mathscr{A}$, which coincides with the gluing of sections for $\omega_C^{\otimes -k}$ given by the condition on the residues in Lemma 1.9.1. Hence one has $\mathscr{A} \cong \omega_C^{\otimes -k}$ in this case as well. The assertion therefore follows. □

As a consequence of Lemma 2.6.1, we record here the following statement, which will be used throughout.

Lemma 2.6.2. *Let $(C, P_\bullet)$ be a smooth n-pointed curve. One has:*

$$H^0(C \setminus P_\bullet, \mathcal{V}_C) \cong H^0(C \setminus P_\bullet, \mathrm{gr}_\bullet \mathcal{V}_C).$$

Proof. We claim that on the affine open set $C \setminus P_\bullet$, one has

$$\mathcal{V}_{\le k} \cong \oplus_{i\le k}\mathcal{V}_{\le i}/\mathcal{V}_{\le i-1} = \mathrm{gr}_{\le k}\mathcal{V}_C. \tag{12}$$

Assuming (12), then one has, for every $k \in \mathbb{Z}_{\ge 0}$, an injection

$$\phi_k \colon \mathrm{gr}_{\le k}\mathcal{V}_C \hookrightarrow \mathcal{V}_C,$$

altogether defining a map $\phi\colon \mathrm{gr}_\bullet\mathcal{V}_C \longrightarrow \mathcal{V}_C$. The map ϕ gives the isomorphism we seek. To see that ϕ is injective, note that any element x in $\mathrm{gr}_\bullet\mathcal{V}_C$ is in fact a finite sum, and hence x is in $\mathrm{gr}_{\le k}\mathcal{V}$ for some k. So if x is mapped to zero by ϕ, then x is mapped to zero by ϕ_k for some k. Since all maps ϕ_k are injective, x is zero. Surjectivity of ϕ follows from the fact that $\mathcal{V}_C$ is filtered by the $\mathcal{V}_{\le k}$.

We prove (12) by induction on k with base case $k = 0$. Lemma 2.6.1 implies that $\mathrm{gr}_k\mathcal{V}_C$ is locally free. On affines, locally free sheaves are projective, and hence on the affine open set $C \setminus P_\bullet$, the following sequence splits:

$$0 \to \mathcal{V}_{\le k-1} \to \mathcal{V}_{\le k} \to \mathrm{gr}_k\mathcal{V}_C \to 0.$$

In particular, on $C \setminus P_\bullet$

$$\mathcal{V}_{\le k} \cong \mathcal{V}_{\le k-1} \oplus \mathrm{gr}_k\mathcal{V}_C \cong \mathrm{gr}_{\le k-1}\mathcal{V}_C \oplus \mathrm{gr}_k\mathcal{V}_C,$$

and (12) holds. □

Remark 2.6.3. We remark that Lemma 2.6.2 does not hold in general over nodal curves. This is because, as noted in Remark 2.5.1, the sheaf $\mathcal{V}_{\le k}$ is not quasi-coherent on a curve with singularities.

2.7. **The flat connection.** The sheaf $\mathcal{V}_C$ supports a flat connection

$$\nabla\colon \mathcal{V}_C \to \mathcal{V}_C \otimes \omega_C$$

defined for smooth curves in [FBZ, §6]. On a smooth open set U in C admitting a global coordinate t (e.g., an open U admitting an étale map $U \to \mathbb{A}^1$), one has a trivialization $\mathcal{V}|_U \simeq_t V \times U$. On $\mathcal{V}|_U$, the connection ∇ is given by $L_{-1} \otimes \mathrm{id}_U + \mathrm{id}_V \otimes \partial_t$ (compare with (2)).

When the curve C is nodal, then the connection is defined as follows. Assume for simplicity that C has only one node Q and let $\nu\colon \widetilde{C} \to C$ be the normalization. Recall that the sheaf $\mathcal{V}_C$ is defined via a subsheaf of the sheaf $\widetilde{\mathcal{V}}$ in (11). One first defines a flat logarithmic connection on the sheaf

$$\oplus_{k\ge 0}\, V_k \otimes \mathscr{O}_{\widetilde{C}}(-k\, Q_+ - k\, Q_-) \otimes \pi_*\mathscr{O}_{\mathscr{A}ut_{\widetilde{C}}}. \tag{13}$$

For this, on a smooth open set of $\widetilde{C}$ not containing the points $Q_\pm$ and admitting a global coordinate t, the connection is given by $L_{-1} \otimes \mathrm{id}_U + \mathrm{id}_V \otimes \partial_t$, as in the smooth case. On each of the local disks $D_{Q_\pm} = \mathrm{Spec}\,\widehat{\mathscr{O}}_{Q_\pm}$ about the points $Q_\pm$, the sheaf in (13) has a trivialization

$$\left(\oplus_{k\ge 0}\, V_k \otimes s_\pm^k\, \mathbb{C}[\![s_\pm]\!]\right) \times D_{Q_\pm}.$$

Here the logarithmic connection is given by $L_{-1}\, s_\pm \otimes \mathrm{id}_D + \mathrm{id}_{V[\![s_\pm]\!]} \otimes s_\pm\, \partial_{s_\pm}$, where $D = D_{Q_\pm}$. As $s_\pm$ vanishes at $Q_\pm$, the connection fixes the stalks at

$Q_\pm$. The connection thus defined on (13) induces a connection on (11), and after gluing the stalks at $Q_\pm$ as in §2.5, it induces a connection on $\mathcal{V}_C$.

2.8. **A coordinate-free view of the Lie algebra ancillary to V.** As a first application of the sheaf of vertex algebras, one obtains a coordinate-free version of the Lie algebra ancillary to V. Namely, for a smooth point P in an algebraic curve C and a formal coordinate t at P, one has

$$H^0\left(D_P^\times, \mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right) \xrightarrow{\simeq_t} \mathfrak{L}_t(V), \tag{14}$$

where, as before, $D_P^\times$ is the punctured formal disk about P on C. The map is given as follows: a section of $\mathcal{V}_C \otimes \omega_C$ on $D_P^\times$ with respect to the t-trivialization

$$B \otimes \sum_{i \geq i_0} a_i t^i dt \quad \in \quad V \otimes_{\mathbb{C}} \mathbb{C}((t)) \otimes_{\mathbb{C}((t))} \mathbb{C}((t))dt \ \simeq_t H^0\big(D_P^\times, \mathcal{V}_C \otimes \omega_C\big)$$

maps to

$$\mathrm{Res}_{t=0}\, Y[B,t] \sum_{i \geq i_0} a_i t^i dt \quad \in \quad \mathfrak{L}_t(V),$$

where $Y[B,t] := \sum_{i\in\mathbb{Z}} B_{[i]} t^{-i-1}$. Sections in $\mathrm{Im}\nabla \subset \mathcal{V}_C \otimes \omega_C$ map to zero, hence this defines a linear map from sections of $\mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla$ on $D_P^\times$ to $\mathfrak{L}_t(V)$. Moreover, the vector space $H^0\left(D_P^\times, \mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right)$ has the structure of a Lie algebra such that (14) is indeed an isomorphism of Lie algebras [FBZ, §§19.4.14, 6.6.9].

3. The chiral Lie algebra

For a stable n-pointed curve $(C, P_\bullet)$ and a vertex operator algebra V, we define the chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ (§3.1), describe the maps to the Lie algebras ancillary to V at P_i (§3.2), and give an explicit description for the chiral Lie algebra on a nodal curve in terms of the normalization of the curve (§3.3). We note that the chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ defined here depends on the sheaf $\mathcal{V}_C$, in contrast to the sheaf $\mathscr{L}_{C\setminus P_\bullet}(V)$ defined in [**?**] which was dependent on $\mathscr{V}_C$. We conclude with a consequence of Riemann-Roch (§3.4).

3.1. **Definition of the chiral Lie algebra.** For $(C, P_\bullet)$ a stable n-pointed curve and V a vertex operator algebra, set

$$\mathcal{L}_{C\setminus P_\bullet}(V) := H^0\left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right).$$

Here $\mathcal{V}_C$ and its flat connection ∇ are as in §2.

3.2. **The chiral Lie algebra maps to the Lie algebra ancillary to V.** For each i, let t_i be a formal coordinate at P_i, let $D_{P_i}^\times$ be the punctured formal disk about P_i on C, and $\mathfrak{L}_{t_i}(V)$ be the Lie algebra ancillary to V (§1.3). Consider the linear map obtained as the composition

$$\mathcal{L}_{C\setminus P_\bullet}(V) \to H^0\left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right) \xrightarrow{\cong} \mathfrak{L}_{t_i}(V). \tag{15}$$

The first map is canonical and obtained by restricting sections. The second map is the isomorphism of Lie algebras (14) and depends on the formal coordinates t_i. From (15), we construct the linear map

$$\varphi_{\mathcal{L}} \colon \mathcal{L}_{C\setminus P_\bullet}(V) \to \oplus_{i=1}^n H^0\left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right) \xrightarrow{\cong} \oplus_{i=1}^n \mathfrak{L}_{t_i}(V). \tag{16}$$

After [FBZ, §19.4.14], the first map is a homomorphism of Lie algebras, hence so is $\varphi_{\mathcal{L}}$, denoted simply φ. The map φ thus induces an action of $\mathcal{L}_{C\setminus P_\bullet}(V)$ on $\mathfrak{L}(V)^{\oplus n}$-modules. This will be used in §4.

3.3. **A close look at the chiral Lie algebra for nodal curves.** Let $(C, P_\bullet)$ be a stable n-pointed curve such that $C \setminus P_\bullet$ is affine. Assume for simplicity that C has exactly one node, which we denote by Q. Let $\widetilde{C} \to C$ be the normalization of C, let Q_+ and Q_- be the two preimages of Q, and set $Q_\bullet = (Q_+, Q_-)$. Let s_+ and s_- be formal coordinates at Q_+ and Q_-, respectively, such that locally around Q, the curve C is given by the equation $s_+s_- = 0$. The chiral Lie algebra for $\left(\widetilde{C}, P_\bullet \sqcup Q_\bullet\right)$ is

$$\mathcal{L}_{\widetilde{C}\setminus P_\bullet\sqcup Q_\bullet}(V) = H^0\left(\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet, \mathcal{V}_{\widetilde{C}} \otimes \omega_{\widetilde{C}}/\mathrm{Im}\,\nabla\right),$$

and consider the linear map given by restriction:

$$\mathcal{L}_{\widetilde{C}\setminus P_\bullet\sqcup Q_\bullet}(V) \to H^0\left(D_{Q_+}^\times, \mathcal{V}_{\widetilde{C}} \otimes \omega_{\widetilde{C}}/\mathrm{Im}\,\nabla\right) \xrightarrow{\simeq_{s_+}} \mathfrak{L}_{Q_+}(V). \tag{17}$$

Recall the triangular decomposition of $\mathfrak{L}_{Q_\pm}(V)$ from (4):

$$\mathfrak{L}_{Q_\pm}(V) = \mathfrak{L}_{Q_\pm}(V)_{<0} \oplus \mathfrak{L}_{Q_\pm}(V)_0 \oplus \mathfrak{L}_{Q_\pm}(V)_{>0}.$$

Let $\sigma_{Q_\pm} \in \mathfrak{L}_{Q_\pm}(V)$ be the image of $\sigma \in \mathcal{L}_{\widetilde{C}\setminus P_\bullet\sqcup Q_\bullet}(V)$, and let $\left[\sigma_{Q_\pm}\right]_0$ be the image of $\sigma_{Q_\pm}$ under the projection $\mathfrak{L}_{Q_\pm}(V) \to \mathfrak{L}_{Q_\pm}(V)_0$.

Recall the involution ϑ of $\mathfrak{L}(V)$ in (6). It restricts to an involution on $\mathfrak{L}(V)_0$ given for homogeneous $B \in V_k$ by

$$\vartheta\left(B_{[k-1]}\right) = (-1)^{k-1}\sum_{i\geq 0}\frac{1}{i!}L_1^i B_{[k-i-1]}.$$

Proposition 3.3.1. *For $C \setminus P_\bullet$ affine, one has*

$$\mathcal{L}_{C\setminus P_\bullet}(V) = \left\{ \sigma \in \mathcal{L}_{\widetilde{C}\setminus P_\bullet\sqcup Q_\bullet}(V) \;\middle|\; \begin{array}{l} \sigma_{Q_+}, \sigma_{Q_-} \in \mathfrak{L}(V)_{\leq 0}, \\ \text{and } \left[\sigma_{Q_-}\right]_0 = \vartheta\left(\left[\sigma_{Q_+}\right]_0\right) \end{array} \right\}.$$

Proof. Since $C \setminus P_\bullet$ is affine, one has

$$\mathcal{L}_{C\setminus P_\bullet}(V) = H^0\left(C\setminus P_\bullet, \mathcal{V}_C \otimes \omega_C\right)/\nabla H^0\left(C \setminus P_\bullet, \mathcal{V}_C\right).$$

To study elements in $\mathcal{L}_{C\setminus P_\bullet}(V)$, we can thus describe their lifts in the vector space $H^0\left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C\right)$. By definition of $\mathcal{V}_C$ over nodal curves via the sheaf $\widetilde{\mathcal{V}}$ in (11) and by Lemma 1.9.1, we have that

$$H^0\left(C\setminus P_\bullet, \mathcal{V}_C \otimes \omega_C\right) \subseteq H^0\left(\widetilde{C}\setminus P_\bullet, \widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-)\right). \tag{18}$$

As in Lemma 2.6.2, sections of $\widetilde{\mathcal{V}}$ and $\mathrm{gr}_\bullet\widetilde{\mathcal{V}}$ on $\widetilde{C}\setminus P_\bullet$ are isomorphic. An argument similar to the one for Lemma 2.6.1 shows that

$$\mathrm{gr}_\bullet\widetilde{\mathcal{V}} \cong \oplus_{k\geq 0}\left(\omega_{\widetilde{C}}(Q_+ + Q_-)^{\otimes -k}\right)^{\oplus \dim V_k}.$$

Hence we have

$$H^0\left(\widetilde{C}\setminus P_\bullet, \widetilde{\mathcal{V}}\right) = \bigoplus_{k\geq 0} H^0\left(\widetilde{C}\setminus P_\bullet, \left(\omega_{\widetilde{C}}(Q_+ + Q_-)^{\otimes -k}\right)^{\oplus \dim V_k}\right).$$

Using this in (18), we deduce that $H^0\left(C\setminus P_\bullet, \mathcal{V}_C\otimes\omega_C\right)$ is identified with a subset of

$$\bigoplus_{k\geq 0} H^0\left(\widetilde{C}\setminus P_\bullet, \left(\omega_{\widetilde{C}}(Q_+ + Q_-)^{\otimes 1-k}\right)^{\oplus \dim V_k}\right) \tag{19}$$

characterized by the constraints given by the gluing conditions.

Step 1. First, we verify the order constraints. Consider the composition of the linear maps

$$\begin{array}{ccc} \bigoplus_{k\geq 0} H^0\left(\widetilde{C}\setminus P_\bullet, \left(\omega_{\widetilde{C}}(Q_+ + Q_-)^{1-k}\right)^{\oplus \dim V_k}\right) & \dashrightarrow & \bigoplus_{k\geq 0} V_k\otimes_{\mathbb{C}} s_\pm^{k-1}\mathbb{C}[\![s_\pm]\!]\,ds_\pm \\ \downarrow & \nearrow_{\simeq_{s_\pm}} & \\ \bigoplus_{k\geq 0} H^0\left(D_{Q_\pm}, \left(\omega_{\widetilde{C}}(Q_+ + Q_-)^{1-k}\right)^{\oplus \dim V_k}\right) & & \end{array} \tag{20}$$

where the left vertical map is the restriction of sections, followed by the $s_\pm$-trivialization. Recall that the image of an element $B\otimes\mu\in V\otimes\mathbb{C}(\!(s_\pm)\!)ds_\pm$ via the projection $V\otimes\mathbb{C}(\!(s_\pm)\!)ds_\pm\to\mathfrak{L}_{Q_\pm}(V)$ is

$$\mathrm{Res}_{s_\pm=0}\, Y\left[B, s_\pm\right]\mu \qquad \in \qquad \mathfrak{L}_{Q_\pm}(V).$$

From the diagram, we can immediately deduce that the image of $\sigma_{Q_\pm}$ inside $\mathfrak{L}_{Q_\pm}(V)$, which we still denote by $\sigma_{Q_\pm}$, lies in $\mathfrak{L}_{Q_\pm}(V)_{\leq 0}\cong\mathfrak{L}(V)_{\leq 0}$, as claimed.

Step 2. Finally, we show how the gluing isomorphism identifying elements of $H^0\left(C\setminus P_\bullet, \mathcal{V}_C\otimes\omega_C\right)$ inside (19) implies $\left[\sigma_{Q_-}\right]_0 = \vartheta\left(\left[\sigma_{Q_+}\right]_0\right)$. For a section σ in (19), $\left[\sigma_{Q_\pm}\right]_0$ denotes the image of σ via the composition of (20) with the projection

$$\oplus_{k\geq 0} V_k\otimes_{\mathbb{C}} s_\pm^{k-1}\mathbb{C}[\![s_\pm]\!]\,ds_\pm \twoheadrightarrow \oplus_{k\geq 0} V_k\otimes_{\mathbb{C}}\mathbb{C}s_\pm^{k-1}ds_\pm.$$

The elements $\left[\sigma_{Q_\pm}\right]_0$ are the restriction of σ at the stalks at $Q_\pm$. For σ to correspond to a section of the left-hand side of (18), the two elements $\left[\sigma_{Q_\pm}\right]_0$ need to satisfy an identity coming from the gluing isomorphism between the stalks at $Q_\pm$.

The gluing isomorphism identifying stalks at $Q_\pm$ of $\widetilde{\mathcal{V}}$ from §2.5 induces the following gluing isomorphism of stalks at $Q_\pm$ of (19):

$$\begin{array}{ccc}
\bigoplus_{k\geq 0}\left(\mathbb{C}s_+^k ds_+^{-k}\right)^{\oplus \dim V_k}\otimes_{\mathbb{C}}\mathbb{C}s_+^{-1}ds_+ & \xrightarrow{\cong} & \bigoplus_{k\geq 0}\left(\mathbb{C}s_-^k ds_-^{-k}\right)^{\oplus \dim V_k}\otimes_{\mathbb{C}}\mathbb{C}s_-^{-1}ds_- \\
\wr\Vert\uparrow & & \downarrow\Vert\wr \\
V\otimes_{\mathbb{C}}\mathbb{C}s_+^{-1}ds_+ & & V\otimes_{\mathbb{C}}\mathbb{C}s_-^{-1}ds_- \\
\cup & & \cup \\
A\otimes s_+^{-1}ds_+ & \longmapsto & -\left(e^{L_1}(-1)^{L_0}A\right)\otimes s_-^{-1}ds_-.
\end{array}$$

The isomorphism between $V\otimes_{\mathbb{C}}\mathbb{C}s_\pm^{-1}ds_\pm$ and the stalk of $\mathcal{L}_{\widetilde{C}\setminus P_\bullet\sqcup Q_\bullet}(V)$ at $Q_\pm$ maps $L_1^i A\otimes s_\pm^{-1}ds_\pm$ to $A_{[k-i-1]}$, for $A\in V_k$. It follows that

$$\left[\sigma_{Q_+}\right]_0 = -e^{L_1}(-1)^{L_0}\left[\sigma_{Q_-}\right]_0 = \vartheta\left(\left[\sigma_{Q_-}\right]_0\right),$$

hence the statement. □

3.4. **A consequence of Riemann-Roch for chiral Lie algebras.** In parallel with §1.10, we have the following statement for chiral Lie algebras. Let C be a smooth curve, possibly disconnected, with two non-empty sets of distinct marked points $P_\bullet = (P_1,\dots,P_n)$ and $Q_\bullet = (Q_1,\dots,Q_m)$. For each $i\in\{1,\dots,m\}$, let s_i be a formal coordinate at the point Q_i. For a section $\sigma\in\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)$, let σ_{Q_i} be the image of σ under the map given by restriction

$$\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)\to H^0\left(D_{Q_i}^\times, \mathcal{V}_C\otimes\omega_C/\mathrm{Im}\,\nabla\right)\xrightarrow{\simeq_{s_i}}\mathfrak{L}_{Q_i}(V).$$

For an integer N, consider

$$\mathfrak{L}_{Q_i}(V, NQ_i) = V\otimes s_i^N\mathbb{C}[\![s_i]\!]/\mathrm{Im}\,\partial.$$

This is a Lie subalgebra of $\mathfrak{L}_{Q_i}(V)$.

Proposition 3.4.1. *Assume that $C\setminus P_\bullet$ is affine. Fix a homogeneous element $E\in V$, and integers d and N. There exists $\sigma\in\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)$ such that:*

$$\begin{aligned}
\sigma_{Q_i} &\equiv E_{[d]} &&\in &&\mathfrak{L}_{Q_i}(V)/\mathfrak{L}_{Q_i}(V,NQ_i), &&\text{for a fixed } i,\\
\sigma_{Q_j} &\equiv 0 &&\in &&\mathfrak{L}_{Q_j}(V)/\mathfrak{L}_{Q_j}(V,NQ_j), &&\text{for all } j\neq i.
\end{aligned}$$

Proof. Since $C\setminus P_\bullet$ is affine, so is $C\setminus P_\bullet\sqcup Q_\bullet$. As in Proof of Proposition 3.3.1, elements of $\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)$ can be lifted to sections of $\mathcal{V}_C\otimes\omega_C$ on $C\setminus P_\bullet\sqcup Q_\bullet$, and thus described as sections of $\oplus_{k\geq 0}\left(\omega_C^{\otimes 1-k}\right)^{\oplus\dim V_k}$ on $C\setminus P_\bullet\sqcup Q_\bullet$ via the isomorphism (18). The statement thus follows from the analogous property of sections of tensor products of ω_C, discussed in §1.10. □

4. Spaces of coinvariants

Given a stable pointed curve $(C, P_\bullet)$ and a vertex operator algebra V, we define spaces of coinvariants using representations of the chiral Lie algebra.

4.1. **Representations of the chiral Lie algebra.** We begin by defining the action of the chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ on $M^\bullet := M^1 \otimes \cdots \otimes M^n$, for V-modules $M^1, \ldots, M^n$. For each i, let t_i be a formal coordinate at P_i, and $\mathfrak{L}_{t_i}(V)$ be the Lie algebra ancillary to V (§1.3). Each $\mathfrak{L}_{t_i}(V)$ acts on the V-module M^i as in §1.3, and the sum $\oplus_{i=1}^n \mathfrak{L}_{t_i}(V)$ acts diagonally on $M^\bullet$. The map (16) thus induces an action of $\mathcal{L}_{C\setminus P_\bullet}(V)$ on $M^\bullet$ as follows: for $\sigma \in \mathcal{L}_{C\setminus P_\bullet}(V)$ and $A^i \in M^i$, one has

$$\sigma\left(A^1 \otimes \cdots \otimes A^n\right) = \textstyle\sum_{i=1}^n A^1 \otimes \cdots \otimes \sigma_{P_i}\left(A^i\right) \otimes \cdots \otimes A^n,$$

where σ_{P_i} is the restriction of the section σ to the punctured formal disk $D^\times_{P_i}$ about P_i on C.

4.2. **Coinvariants.** When $C \setminus P_\bullet$ is affine, the *space of coinvariants* at $(C, P_\bullet, t_\bullet)$ is

$$\mathbb{V}(V; M^\bullet)_{(C,P_\bullet,t_\bullet)} := M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V)} = M^\bullet / \mathcal{L}_{C\setminus P_\bullet}(V) \cdot M^\bullet.$$

This is the largest quotient of $M^\bullet$ on which $\mathcal{L}_{C\setminus P_\bullet}(V)$ acts trivially. In general, when $C \setminus P_\bullet$ is not necessarily affine, the *space of coinvariants* at $(C, P_\bullet, t_\bullet)$ is defined as the direct limit

$$\mathbb{V}(V; M^\bullet)_{(C,P_\bullet,t_\bullet)} := \varinjlim_{(Q_\bullet, s_\bullet)} \mathbb{V}(V; M^\bullet \sqcup (V, \ldots, V))_{(C, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)} \tag{21}$$

where $Q_\bullet = (Q_1, \ldots, Q_m)$ ranges over the set of stable points of C such that $P_\bullet \cap Q_\bullet = \emptyset$ and $C \setminus P_\bullet \sqcup Q_\bullet$ is affine, and $s_\bullet = (s_1, \ldots, s_m)$, with s_i a formal coordinate at Q_i, for each i. The above direct limit is well defined thanks to propagation of vacua (this is similar to the case of affine Lie algebras [Fak, Loo], see [DGT1] for more details).

The vector spaces $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet,t_\bullet)}$ were initially considered for stable pointed curves and modules of Virasoro algebras in [BFM], generalized to conformal vertex algebras and smooth pointed curves in [FBZ, BD], and extended to stable pointed curves in [DGT1]. Since they are *independent of coordinates*, one can define vector spaces of coinvariants, denoted $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)}$, non-canonically isomorphic to $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet,t_\bullet)}$ for each $t_\bullet$ (see [DGT1] for more details).

Remark 4.2.1. In §5 we show that the vector spaces of coinvariants are finite-dimensional under some natural hypotheses. In particular, one can detect many aspects of the vector spaces of coinvariants by studying their dual spaces, referred to in the literature as vector spaces of *conformal blocks*. Conformal blocks are often studied through their identification with systems of correlation functions [TUY, §2.4], [TK, §5], [NT, §5.3], [FZ, §2.3], [FBZ, §§4.5, 10.3], and [Zhu2, §6]. For instance, systems of correlation functions were used in the proofs of *propagation of vacua* and the *factorization property* for conformal blocks defined by modules over affine Lie algebras [TUY], in [NT, Thms 5.6.1 and 8.4.3] for conformal blocks in genus zero defined by modules over vertex algebras, and in [Zhu1, Thm 6.1], and [FBZ, §10.3.1] to prove propagation of vacua for conformal blocks in positive genus defined by modules over vertex algebras. In [FZ, Thms 2.3.3 and 2.3.4] correlation functions are used to construct vertex algebras and their representations. While we only nominally use this identification in Remark 6.2.2 and Proposition 6.2.1 in service of the proof of the Factorization Theorem, it is natural to ask whether, as for those associated to vertex algebras induced from affine Lie algebras, the Heisenberg Lie algebra, and the Virasoro algebra [KZ], [MMS], systems of correlation functions are governed by interesting differential equations.

5. Finite-dimensionality of coinvariants

Using coinvariants by the action of Zhu's Lie algebra (§A), Abe and Nagatomo show that spaces of coinvariants at smooth pointed curves of arbitrary genus are finite-dimensional [AN]. To be well-defined, Zhu's Lie algebra requires that the vertex algebra is quasi-primary generated. The chiral Lie algebra provides a generalization of Zhu's Lie algebra beyond this constraint. We show here that the result of [AN] extends to coinvariants by the action of the chiral Lie algebra. Moreover, we further extend the result in [AN] by allowing the following twist of the chiral Lie algebra: given a smooth n-pointed curve $(C, P_\bullet)$, and an effective divisor $D=\sum_{i=1}^m n_i Q_i$ on C not supported at $P_\bullet$, consider

$$\mathcal{L}_{C\setminus P_\bullet}(V,D) := H^0\left(C\setminus P_\bullet, \mathcal{V}_C\otimes\omega_C(-D)/\mathrm{Im}\nabla\right), \tag{22}$$

where $\mathrm{Im}\nabla$ denotes the restriction of $\nabla(\mathcal{V}_C)$ to $\mathcal{V}_C\otimes\omega_C(-D)$. This is the space of sections in $\mathcal{L}_{C\setminus P_\bullet}(V)$ vanishing with order at least n_i at Q_i, for each i, and gives a Lie subalgebra of $\mathcal{L}_{C\setminus P_\bullet}(V)$. To state the result, we review the definition of C_2-cofiniteness.

5.1. C_2**-cofiniteness.** Let M be a V-module (e.g., $M=V$) and consider the following subset of M for $k\geq 2$:

$$C_k(M) := \mathrm{span}_{\mathbb{C}}\left\{A_{(-k)}m \,:\, A\in V,\, m\in M\right\}.$$

One says that M is C_k*-cofinite* if $\dim_{\mathbb{C}} M/C_k(M)<\infty$. If $V=\oplus_{i\geq 0}V_i$ is a C_2-cofinite vertex operator algebra with $V_0\simeq\mathbb{C}$, then any finitely generated V-module is C_k-cofinite, for $k\geq 2$ [Buh]. As explained in [Ara1], the C_2-cofiniteness has a natural geometric interpretation which generalizes the concept of lisse modules introduced in [BFM] for the Virasoro algebra.

Proposition 5.1.1. *Let $V=\oplus_{i\geq 0}V_i$ be a C_2-cofinite vertex operator algebra with $V_0\cong\mathbb{C}$. Let C be a smooth curve with distinct points $P_1,\dots P_n$, and D an effective divisor on C not supported at $P_\bullet$. For finitely generated V-modules $M^1,\dots,M^n$, the coinvariants $M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}$ are finite-dimensional.*

Proof. Fix formal coordinates t_i at P_i, for each i, and recall the map from (15): $\mathcal{L}_{C\setminus P_\bullet}(V,D)\to\mathfrak{L}_{P_i}(V)$, $\sigma\mapsto\sigma_{P_i}$. Define a filtration on $\mathcal{L}_{C\setminus P_\bullet}(V,D)$ by

$$\mathcal{F}_k\,\mathcal{L}_{C\setminus P_\bullet}(V,D) := \left\{\sigma\in\mathcal{L}_{C\setminus P_\bullet}(V,D)\,|\,\deg\sigma_{P_i}\leq k, \text{ for all } i\right\}$$

for $k \in \mathbb{N}$, giving $\mathcal{L}_{C\setminus P_\bullet}(V,D)$ the structure of a filtered Lie algebra. Consider also the filtration on $M^\bullet$ given by

$$\mathcal{F}_k M^\bullet = \bigoplus_{0\le d\le k} M^\bullet_d, \quad \text{where} \quad M^\bullet_d := \sum_{d_1+\cdots+d_n=d} M^1_{d_1}\otimes\cdots\otimes M^n_{d_n}.$$

Since $\mathcal{F}_k\,\mathcal{L}_{C\setminus P_\bullet}(V,D)\cdot \mathcal{F}_l M^\bullet \subset \mathcal{F}_{k+l}M^\bullet$, the $\mathcal{L}_{C\setminus P_\bullet}(V,D)$-module $M^\bullet$ is a filtered $\mathcal{L}_{C\setminus P_\bullet}(V,D)$-module. Finally, one has an induced filtration on $M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}$:

$$\mathcal{F}_k\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}\right) := \left(\mathcal{F}_k M^\bullet + \mathcal{L}_{C\setminus P_\bullet}(V,D)\cdot M^\bullet\right)/\mathcal{L}_{C\setminus P_\bullet}(V,D)\cdot M^\bullet.$$

Step 1. Let U be a finite-dimensional subspace of V such that $V = U\oplus C_2(V)$. Contrary to [AN], elements of U are not required to be quasi-primary here. Let d_U be the maximum of the degree of the homogeneous elements in U. Similar to [AN, Lemma 4.1], by an application of the Riemann-Roch and the Weierstrass gap theorem, there exists an integer N such that

$$H^0\left(C, \omega_C^{\otimes 1-k}\left(lP_i - D\right)\right) \neq \emptyset, \quad \text{for all } k\le d_U,\ l\ge N,\ i\in\{1,\dots,n\}.$$

Step 2. For a V-module M and with N as in Step 1, define the subset

$$C_N(U,M) = \operatorname{span}_{\mathbb{C}}\left\{A_{(-l)}m \,:\, A\in U,\, m\in M,\, l\ge N\right\}.$$

We claim that for each i the set $M^1\otimes\cdots\otimes C_N\left(U,M^i\right)\otimes\cdots\otimes M^n$ is in the kernel of the canonical surjective linear map

$$M^\bullet \overset{\pi}{\twoheadrightarrow} \operatorname{gr}_\bullet\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}\right) := \underset{k\ge 0}{\oplus}\,\mathcal{F}_k\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}\right)/\mathcal{F}_{k-1}\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}\right).$$

For this, it is enough to show that $\pi\left(m_1\otimes\cdots\otimes A_{(-l)}m_i\otimes\cdots\otimes m_n\right) = 0$, for homogeneous $A\in U$ of degree k, $m_i\in M^i_{d_i}$, and $l\ge N$. Note that $C\setminus P_i$ is affine, for all i. As in the proof of Proposition 3.3.1, elements of $\mathcal{L}_{C\setminus P_i}(V,D)\subset\mathcal{L}_{C\setminus P_\bullet}(V)$ can be lifted to sections of $\mathcal{V}_C\otimes\omega_C(-D)$ on $C\setminus P_i$. By Lemmas 2.6.2 and 2.6.1, the vector space of such sections is isomorphic to the space of sections of

$$\oplus_{k\ge 0}\, V_k\otimes\omega_C^{\otimes 1-k}(-D) \tag{23}$$

on $C\setminus P_i$. Following Step 1, there exists a section $\sigma = A\otimes\mu$ of (23) on $C\setminus P_i$ such that its image via the map $\mathcal{L}_{C\setminus P_i}(V,D)\to\mathfrak{L}_{P_i}(V)$ from (15) is

$$\sigma_{P_i} = A_{[-l]} + \sum_{j>-l} c_j A_{[j]}, \quad \text{for some } c_j\in\mathbb{C}.$$

One has $A_{[-l]}\cdot M^i_{d_i}\subset M^i_{d_i+k+l-1}$ and $A_{[j]}\cdot M^i_{d_i}\subset M^i_{d_i+k+l-2}$ for $j>-l$. Moreover, since μ is holomorphic at a point $P_j\neq P_i$, one has $\sigma_{P_j} = \sum_{p\ge 0} b_p B_{[p]}$, for some $b_p\in\mathbb{C}$ and $B\in V$; such vector B is obtained from A by the action of an element in $\operatorname{Aut}\mathcal{O}$ producing the isomorphism between the stalks of $\mathcal{V}_C$ at P_i and at P_j, hence $B\in V_{\le k}$. It follows that $\sigma_{P_j}\cdot M^j_{d_j}\subset M^j_{d_j+k-1}$. From the identity

$$\sigma\left(m_1\otimes\cdots\otimes m_n\right) = \textstyle\sum_{j=1}^n m_1\otimes\cdots\otimes\sigma_{P_j}(m_j)\otimes\cdots\otimes m_n,$$

one has

$$m_1 \otimes \cdots \otimes A_{(-l)} m_i \otimes \cdots \otimes m_n \in \mathcal{F}_{\sum_j d_j + k + l - 2} M^\bullet + \mathcal{L}_{C \setminus P_\bullet}(V, D) \cdot M^\bullet.$$

Since the element on the left-hand side is in $\mathcal{F}_{\sum_j d_j + k + l - 1} M^\bullet$, it follows that it maps to zero via π. The claim follows.

Step 3. After Step 2, the map π factors through

$$M^1/C_N\left(U, M^1\right) \otimes \cdots \otimes M^n/C_N\left(U, M^n\right) \stackrel{\pi}{\twoheadrightarrow} \mathrm{gr}_\bullet\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,D)}\right). \tag{24}$$

By [AN, Prop. 4.5], there is a positive integer k such that $C_k\left(M^i\right) \subset C_N\left(U, M^i\right)$ for all i. In particular, $\dim M^i/C_N\left(U, M^i\right) < \dim M^i/C_k\left(M^i\right)$. These are finite as the M^i are all C_k-cofinite by [Buh]. It follows that the source in (24) is finite-dimensional, hence so is the target. This implies that the coinvariants are finite-dimensional as well. □

6. The modules Z and $\overline{Z}$

We introduce here two modules which will aide the proof of the factorization property of spaces of coinvariants. In §6.1 we define the modules Z and $\overline{Z}$ and give their properties, and in §6.2 we study spaces of coinvariants with a module $\overline{Z}$.

6.1. **Definitions and properties.** Let V be a vertex operator algebra. Recall the associative algebra $\mathscr{U}(V)$ from §1.4. Consider the $\mathscr{U}(V)^{\otimes 2}$-module

$$Z := \left(\mathrm{Ind}_{\mathscr{U}(V)_{\leq 0}}^{\mathscr{U}(V)} A(V)\right)^{\otimes 2} = \left(\mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} A(V)\right)^{\otimes 2}$$

where $\mathscr{U}(V)_{<0}$ acts trivially on $A(V)$, and the action of $\mathscr{U}(V)_0$ on $A(V)$ is induced from the projection $\mathscr{U}(V)_0 \to A(V)$. With the notation from §1.6, one has that $Z = M(A(V))^{\otimes 2}$, where $M(A(V))$ is the generalized Verma $\mathscr{U}(V)$-module induced from the natural representation $A(V)$ of $A(V)$.

In §7, we will also consider a quotient $\overline{Z}$ of Z. Let $\mathscr{P}$ be the subalgebra of $\mathscr{U}(V)^{\otimes 2}$ generated by $\mathscr{U}(V) \otimes_{\mathbb{C}} \mathscr{U}(V)_{\leq 0}$ and $\mathscr{U}(V)_{\leq 0} \otimes_{\mathbb{C}} \mathscr{U}(V)$. Consider the $\mathscr{U}(V)^{\otimes 2}$-module

$$\overline{Z} := \mathrm{Ind}_{\mathscr{P}}^{\mathscr{U}(V)^{\otimes 2}} A(V) = \left(\mathscr{U}(V)^{\otimes 2}\right) \otimes_{\mathscr{P}} A(V)$$

where $\mathscr{U}(V) \otimes_{\mathbb{C}} \mathscr{U}(V)_{<0}$ and $\mathscr{U}(V)_{<0} \otimes_{\mathbb{C}} \mathscr{U}(V)$ act trivially on $A(V)$, and the action of $\mathscr{U}(V)_0 \otimes_{\mathbb{C}} \mathscr{U}(V)_0$ on $A(V)$ is induced via the natural surjection $\mathscr{U}(V)_0 \otimes_{\mathbb{C}} \mathscr{U}(V)_0 \to A(V) \otimes_{\mathbb{C}} A(V)$ from the action of $A(V) \otimes_{\mathbb{C}} A(V)$ given by

$$(a \otimes b)(c) = a \cdot c \cdot (-\vartheta(b)), \quad \text{for } a \otimes b \in A(V) \otimes A(V),\ c \in A(V).$$

Lemma 6.1.1. *Let V be a rational vertex operator algebra. One has $\mathscr{U}(V)^{\otimes 2}$-module isomorphisms*

$$Z \cong \bigoplus_{W,Y \in \mathscr{W}} \left(W \otimes W_0^\vee\right) \otimes \left(Y \otimes Y_0^\vee\right) \qquad \text{and} \qquad \overline{Z} \cong \bigoplus_{W \in \mathscr{W}} W \otimes W',$$

where $\mathscr{W}$ is the set of all the finitely many simple V-modules.

Proof. Since V is rational, the algebra $A(V)$ is semisimple [Zhu2]. From Wedderburn's theorem, one has $A(V) = \oplus_{E\in\mathscr{E}} E \otimes E^\vee$, where $\mathscr{E}$ is the set of all finitely many simple $A(V)$-modules. Using the one-to-one correspondence between simple V-modules and simple $A(V)$-modules [Zhu2], and rationality of V which implies that the V-module induced from any simple $A(V)$-module is simple, it follows that each simple V-module is $W = \mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E$, for some $E \in \mathscr{E}$. Moreover, there exists a canonical V-module isomorphism $\mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E^\vee \cong (\mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E)'$, for $E \in \mathscr{E}$ [NT, Prop. 7.2.1]. The statement follows by linearity. □

6.2. **Replacing coinvariants with the module Z.** In this section we describe a Lie subalgebra $\mathcal{L}_{C\setminus P_\bullet}(V, \{Q_1, Q_2\})$ of $\mathcal{L}_{C\setminus P_\bullet}(V)$, which provides the following statement, generalizing [NT, Prop. 7.2.2, Cor. 8.6.2] from rational curves to curves of arbitrary genus.

Proposition 6.2.1. *Consider a coordinatized $(n+2)$-pointed smooth curve $(C, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)$, possibly disconnected, such that $C \setminus P_\bullet$ is affine. Let V be a rational vertex operator algebra. Given V-modules $M^1, \dots, M^n$, the map*

$$M^\bullet \to M^\bullet \otimes Z, \qquad w \mapsto w \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)},$$

where $\mathbf{1}^{A(V)} \in A(V)$ is the unit, induces an isomorphism of vector spaces

$$h\colon M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})} \xrightarrow{\cong} (M^\bullet \otimes_{\mathbb{C}} Z)_{\mathcal{L}_{C\setminus P_\bullet \sqcup Q_\bullet}(V)}.$$

The proof of Proposition 6.2.1, given after the definition of the Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V, \{Q_1, Q_2\})$, involves the study of the chiral Lie algebra in §3, Proposition 3.4.1, and an extension of the statement on propagation of vacua given in Remark 6.2.2.

As in the statement, let C be a smooth curve, possibly disconnected, with two nonempty, disjoint sets of distinct marked points $P_\bullet = (P_1, \dots, P_n)$ and $Q_\bullet = (Q_1, Q_2)$. Assume that $C \setminus P_\bullet$ is affine. After Lemmas 2.6.2 and 2.6.1, one has

$$H^0(C \setminus P_\bullet, \mathcal{V}_C) \cong \oplus_{k\geq 0} H^0\left(C \setminus P_\bullet, V_k \otimes_{\mathbb{C}} \omega_C^{\otimes -k}\right).$$

Using this isomorphism, consider the following Lie subalgebra of the chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$:

(25)
$$\mathcal{L}_{C\setminus P_\bullet}(V, \{Q_1, Q_2\}) := \frac{\bigoplus_{k\geq 0} H^0\left(C \setminus P_\bullet, V_k \otimes_{\mathbb{C}} \omega_C^{\otimes 1-k}(-kQ_1 - kQ_2)\right)}{\nabla H^0(C \setminus P_\bullet, \mathcal{V}_C)}.$$

As in (22), $\nabla H^0(C \setminus P_\bullet, \mathcal{V}_C)$ is the intersection of $\mathrm{Im}\nabla$ with the subspace $\bigoplus_{k\geq 0} H^0\left(C \setminus P_\bullet, V_k \otimes_{\mathbb{C}} \omega_C^{\otimes 1-k}(-kQ_1 - kQ_2)\right)$ of $H^0(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C)$.

Set $\mathfrak{L}_{P_\bullet}(V) := \oplus_{i=1}^n \mathfrak{L}_{P_i}(V)$ and $\mathfrak{L}_{Q_\bullet}(V) := \mathfrak{L}_{Q_1}(V) \oplus \mathfrak{L}_{Q_2}(V)$. Fixing formal coordinates t_i at the point P_i and s_i at Q_i, one has Lie algebra homomorphisms

(26) $\quad \mathcal{L}_{C\setminus P_\bullet}(V, \{Q_1, Q_2\}) \to \mathfrak{L}_{P_\bullet}(V)$ and $\mathcal{L}_{C\setminus P_\bullet}(V, \{Q_1, Q_2\}) \to \mathfrak{L}_{Q_\bullet}(V)$.

The Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})$ consists of the elements in $\mathcal{L}_{C\setminus P_\bullet}(V)$ whose image in $\mathfrak{L}_{Q_\bullet}(V)\cong\mathfrak{L}(V)^{\oplus 2}$ via the restriction map in (26) lies in $\mathfrak{L}(V)^{\oplus 2}_{<0}\subset\mathfrak{L}_{Q_\bullet}(V)$. Indeed, the image of an element of $\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})$ in $\mathfrak{L}_{Q_\bullet}(V)$ is a linear combination of elements of type

$$\sum_{i\geq k} a_i A_{[i]}\oplus\sum_{j\geq k} b_j B_{[j]}\quad\in\quad\mathfrak{L}(V)^{\oplus 2}_{<0}$$

for homogeneous $A,B\in V$ of degree $k\geq 0$ and coefficients $a_i,b_j\in\mathbb{C}$.

Remark 6.2.2. To prove Proposition 6.2.1 we are going to use a generalization of the statement on propagation of vacua from [FBZ, Thm 10.3.1], originally stated for affine Lie algebras in [TUY, Prop. 2.2.3], to slightly larger spaces of coinvariants and conformal blocks. Namely, one has a canonical isomorphism of vector spaces

$$\mathrm{Hom}\left((M^\bullet\otimes V^\bullet)_{\mathcal{L}_{C\setminus P_\bullet\sqcup R_\bullet}(V,\{Q_1,Q_2\})},\mathbb{C}\right)\cong\mathrm{Hom}\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})},\mathbb{C}\right)$$

for a set of distinct points $R_\bullet$ disjoint from $P_\bullet\sqcup Q_\bullet$.

For this, let $R\in C\setminus P_\bullet\sqcup Q_\bullet$ and fix a formal coordinate at R. As in [FBZ, §10.3], the natural map

$$\xi\colon\mathrm{Hom}\left((M^\bullet\otimes V)_{\mathcal{L}_{C\setminus P_\bullet\sqcup R}(V,\{Q_1,Q_2\})},\mathbb{C}\right)\to\mathrm{Hom}\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})},\mathbb{C}\right)$$

given by $\xi(\varphi)(w)=\varphi(w\otimes\mathbf{1}^V)$ is an isomorphism. Indeed, using the strong residue theorem [FBZ, Thm 9.2.9], [Tat], we can identify the target of ξ as those linear functionals φ on $M^\bullet$ such that for any $w\in M^\bullet$, the elements $\langle\varphi,\mathcal{Y}^{M^i}(-)(w)\rangle$ of $\bigoplus_{k\geq 0}H^0\left(D^\times_{P_i},V_k^\vee\otimes\omega_C^{\otimes k}(kQ_1+kQ_2)\right)$, with $i=1,\dots,n$, can be extended to the same section φ_w of

$$\oplus_{k\geq 0}H^0\left(C\setminus P_\bullet,V_k^\vee\otimes\omega_C^{\otimes k}(kQ_1+kQ_2)\right).$$

Here $\mathcal{Y}^{M^i}(-)$ denotes the coordinate independent version of $Y^{M^i}(-,z)$ as in [FBZ, §7.3]. Using this section, one realizes the inverse of ξ by

$$\xi^{-1}(\varphi)(w\otimes A)=\varphi_w(A).$$

One checks that this realizes the inverse of ξ as in the proof of [FBZ, §10.3.1]. Iterating on the number of points in $R_\bullet$ gives the assertion.

Proof of Proposition 6.2.1.

Step 1. We first show that the map h is well-defined. Observe that Z is naturally equipped with a left action of $\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)$ induced by the Lie algebra homomorphisms $\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)\to\mathfrak{L}_{Q_\bullet}(V)\to\mathscr{U}(V)^{\otimes 2}$. For $\sigma\in\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})\subset\mathcal{L}_{C\setminus P_\bullet\sqcup Q_\bullet}(V)$, let $\sigma_{P_\bullet}$ be the image of σ in $\mathfrak{L}_{P_\bullet}(V)$ via the restriction map in (26), and similarly let σ_{Q_i} be its image in $\mathfrak{L}_{Q_i}(V)$, for $i=1,2$. Since $\sigma_{Q_i}\in\mathfrak{L}(V)_{<0}$, the elements $\sigma_{Q_1}\otimes 1$ and $1\otimes\sigma_{Q_2}$ act trivially on $A(V) \otimes A(V) \subset Z$. This implies

$$\sigma_{P_\bullet}(w) \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)} = \sigma\left(w \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)}\right) - w \otimes \sigma_{Q_1}\left(\mathbf{1}^{A(V)}\right) \otimes \mathbf{1}^{A(V)} - w \otimes \mathbf{1}^{A(V)} \otimes \sigma_{Q_2}\left(\mathbf{1}^{A(V)}\right) = \sigma\left(w \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)}\right) \tag{27}$$

for $w \in M^\bullet$. It follows that the zero element is mapped to the zero element, hence the map h between the spaces of coinvariants is well-defined.

Step 2. Next, we show that the map h is surjective: given $w \otimes z_1 \otimes z_2$ in $M^\bullet \otimes Z$, there exists $w' \in M^\bullet$ such that

$$w \otimes z_1 \otimes z_2 \equiv w' \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)} \mod \mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)\,(M^\bullet \otimes Z).$$

By linearity, and by reordering elements in $\mathscr{U}(V)$, we can reduce to the case

$$z_1 \otimes z_2 = D_l \cdots D_1 \mathbf{1}^{A(V)} \otimes E_m \cdots E_1 \mathbf{1}^{A(V)},$$

with each D_i and E_j in $\mathfrak{L}(V)_{\geq 0}$. The surjectivity is clear when $l = m = 0$. By induction on l (and similarly on m), it is then enough to show that

$$w \otimes z_1 \otimes z_2 \equiv w' \otimes z_1' \otimes z_2 \qquad \mod \mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z)$$

for some w' in $M^\bullet$, when $z_1 = D_{[d]}\,(z_1')$ for some homogeneous $D \in V$ and $D_{[d]}$ in $\mathfrak{L}(V)_{\geq 0}$. Each component of the curve C has at least one of the marked points in $P_\bullet$. By Proposition 3.4.1, there exists $\sigma \in \mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)$ such that

$$\begin{aligned} \sigma_{Q_1} &\equiv D_{[d]} && \in && \mathfrak{L}_{Q_1}(V)/\mathfrak{L}_{Q_1}(V, NQ_1), \\ \sigma_{Q_2} &\equiv 0 && \in && \mathfrak{L}_{Q_2}(V)/\mathfrak{L}_{Q_2}(V, NQ_2), \end{aligned}$$

for $N >> 0$. It is enough to take N such that both

$$D_{[N]}(z_1') \otimes z_2 \quad \text{and} \quad z_1' \otimes D_N(z_2)$$

are zero in Z. Such N exists because $\mathscr{U}(V)$ acts smoothly on each component of Z. Moreover, σ_{Q_2} is $\sum_{i \geq N} a_i A_{[i]}$, with $a_i \in \mathbb{C}$ and $A \in V$ obtained from D via the action of an element of $\operatorname{Aut} \mathcal{O}$ producing the isomorphism between the stalks of $\mathcal{V}_C$ at Q_1 and at Q_2. It follows that $A \in V_{\leq \deg D}$, hence $\sigma_{Q_2}(z_2) = 0$. This implies

$$\sigma_{Q_1}(z_1') \otimes z_2 + z_1' \otimes \sigma_{Q_2}(z_2) = D_{[d]} \cdot z_1' \otimes z_2 = z_1 \otimes z_2.$$

It follows that

$$w \otimes z_1 \otimes z_2 = \sigma(w \otimes z_1' \otimes z_2) - \sigma_{P_\bullet}(w) \otimes z_1' \otimes z_2,$$

hence

$$w \otimes z_1 \otimes z_2 \equiv -\sigma_{P_\bullet}(w) \otimes z_1' \otimes z_2, \qquad \mod \mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z).$$

Repeating the same argument for z_2, the surjectivity of h follows.

Step 3. Finally, we show that h is injective. Equivalently, we show that the dual map

$$h^\vee \colon \mathrm{Hom}\left((M^\bullet \otimes_{\mathbb{C}} Z)_{\mathcal{L}_{C\setminus P_\bullet \sqcup Q_\bullet}(V)}, \mathbb{C}\right) \to \mathrm{Hom}\left(M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\})}, \mathbb{C}\right)$$

given by

$$h^\vee(\Phi)(w) = \Phi\left(w \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)}\right), \qquad \text{for } w \in M^\bullet,$$

is surjective. Let $\Phi_{0,0}$ be an element in the target of $h^\vee$, i.e., $\Phi_{0,0}$ is a linear functional on $M^\bullet$ vanishing on the subspace $\mathcal{L}_{C\setminus P_\bullet}(V,\{Q_1,Q_2\}) \cdot M^\bullet$. In the following, we construct a linear functional Φ on $M^\bullet \otimes_{\mathbb{C}} Z$ vanishing on $\mathcal{L}_{C\setminus P_\bullet \sqcup Q_\bullet}(V) \cdot (M^\bullet \otimes_{\mathbb{C}} Z)$ such that $h^\vee(\Phi) = \Phi_{0,0}$. The argument uses the formalism of systems of correlation functions, as developed in this setting in [NT, §5] for rational curves, with the extensions to curves of arbitrary genus in [FBZ, §10.3] (see Remark 4.2.1).

Step 3(a). Given $\Phi_{0,0}$, one first constructs linear functionals $\Phi_{l,m}$ on $M^\bullet \otimes V^{\otimes l+m}$ for all $l, m \geq 0$. By induction on l and m, it is enough to construct $\Phi_{l,m}$ starting from $\Phi_{l-1,m}$. This uses a version of propagation of vacua similar to the one proved in [FBZ, §10.3.1]. Given $l, m \in \mathbb{Z}_{\geq 0}$, fix points $R_\bullet = R_{[l]} = (R_1, \ldots, R_l)$ and $S_\bullet = (S_1, \ldots, S_m)$ in $C \setminus P_\bullet \sqcup Q_\bullet$ such that $R_\bullet \sqcup S_\bullet$ is a collection of distinct points. Additionally, fix formal coordinates at the points $R_\bullet \sqcup S_\bullet$. By the generalization of propagation of vacua given in Remark 6.2.2, there is a canonical isomorphism of vector spaces

$$\xi \colon \mathrm{Hom}\left(\left(M^\bullet \otimes_{\mathbb{C}} V^{\otimes l+m}\right)_{\mathcal{L}_{C\setminus P_\bullet \sqcup R_{[l]} \sqcup S_\bullet}(V,\{Q_1,Q_2\})}, \mathbb{C}\right) \to \mathrm{Hom}\left(\left(M^\bullet \otimes_{\mathbb{C}} V^{\otimes l-1+m}\right)_{\mathcal{L}_{C\setminus P_\bullet \sqcup R_{[l-1]} \sqcup S_\bullet}(V,\{Q_1,Q_2\})}, \mathbb{C}\right).$$

One then defines

$$\Phi_{l,m} := \xi^{-1}\left(\Phi_{l-1,m}\right). \tag{28}$$

Observe that we could have equivalently defined $\Phi_{l,m} = \xi^{-1}\left(\Phi_{l,m-1}\right)$ interchanging the role of l and m.

Step 3(b). The linear functional $\Phi_{l,m}$ gives rise to a meromorphic form on C^{l+m} as follows. For fixed l and m, by varying the points $R_\bullet$ and $S_\bullet$ in $C\setminus P_\bullet \sqcup Q_\bullet$, the space of conformal blocks, as in the source of the map ξ above, becomes a fiber of the sheaf of conformal blocks $\mathbb{V}(M^\bullet \otimes V^{l+m})^\vee$ on $C^{l+m} \setminus \Delta \cup \Gamma_{P_\bullet} \cup \Gamma_{Q_\bullet}$. Here Δ is the locus of $(l+m)$-tuples where two components coincide; $\Gamma_{P_\bullet}$ is the locus of $(l+m)$-tuples where one component equals one of the points in $P_\bullet$; and $\Gamma_{Q_\bullet}$ is the locus of $(l+m)$-tuples where one component equals Q_1 or Q_2. Similar sheaves have been studied in [FBZ, DGT1]. The linear functionals $\Phi_{l,m}$ patch together to define a global meromorphic section $\Phi_{l,m(R_\bullet, S_\bullet)}$ of $\mathbb{V}(M^\bullet \otimes V^{l+m})^\vee$.

Now fix an element $w\otimes D^\bullet\otimes E^\bullet \in M\otimes V^{\otimes l+m}$, with $D^\bullet = D^l\otimes\cdots\otimes D^1$ and $E^\bullet = E^m\otimes\cdots\otimes E^1$. As in Remark 6.2.2, the strong residue theorem allows us to attach to such a vector and to the section $\Phi_{l,m(R_\bullet,S_\bullet)}$ a meromorphic form on C^{l+m} denoted by

$$\Phi_{l,m}\left(w\otimes D^\bullet\otimes E^\bullet\right)_{(R_\bullet,S_\bullet)}. \tag{29}$$

When the elements D^i and E^j are homogeneous of degree d_i and e_j, respectively, this form is an element of

$$H^0\left(C^{l+m}\setminus\Delta\cup\Gamma_{P_\bullet}, \omega_C^{\boxtimes_{i=1}^l d_i}\boxtimes\omega_C^{\boxtimes_{j=1}^m e_j}\left(\sum_{i=1}^l d_i\Gamma_{R_i,Q_\bullet}+\sum_{j=1}^m e_j\Gamma_{S_j,Q_\bullet}\right)\right),$$

where $\Gamma_{R_i,Q_\bullet}\subset C^{l+m}$ is the locus where R_i coincides with Q_1 or Q_2, and $\Gamma_{S_j,Q_\bullet}\subset C^{l+m}$ is the locus where S_j coincides with Q_1 or Q_2. Note that the order of the poles along Δ and $\Gamma_{P_\bullet}$ is unbounded.

Step 3(c). In the following, it is enough to consider the restriction of (29) to a rational subset of C^{l+m}. Namely, consider a rational subset $U\subset C\setminus Q_\bullet$ containing the points $P_\bullet$. After removing finitely many points in $C\setminus Q_\bullet$, one can indeed find such a set: consider a ramified cover $C\to\mathbb{P}^1$ and remove enough points to get an étale map $U\to\mathbb{A}^1$. Such a map induces a formal coordinate at every point of U, hence a trivialization of $\mathscr{A}ut_C$ on U [FBZ, §6.5.2]. That is, there exists a global coordinate t on U inducing a formal coordinate $t-u$ at every point $u\in U$. In turn, this gives a trivialization of $\mathcal{V}_C$ on U. In particular, using coordinates $t-x_i$ at R_i and $t-y_j$ at S_j, and given homogeneous E^i and D^j in V of degree d_i and e_j, respectively, we can rewrite the restriction of (29) to U^{l+m} as

$$\overline{\Phi}_{l,m}(w\otimes D^\bullet\otimes E^\bullet)_{(R_\bullet,S_\bullet)}\,(dx_1)^{d_1}\cdots(dx_l)^{d_l}(dy_1)^{e_1}\cdots(dy_m)^{e_m} \tag{30}$$

where $\overline{\Phi}_{l,m}(w\otimes D^\bullet\otimes E^\bullet)_{(R_\bullet,S_\bullet)}$ is a rational function with poles along Δ and $\Gamma_{P_\bullet}$.

Note that we require the points Q_1 and Q_2 to lie outside of U. The formal coordinate s_i at Q_i satisfies $s_i=\rho_i(t)$, for a change of variable $\rho_i\in\mathrm{Aut}\,(\mathcal{O})$, for $i=1,2$. This element ρ_i induces an isomorphism $\mathfrak{L}_t(V)\to\mathfrak{L}_{s_i}(V)$, still denoted ρ_i.

Step 3(d). We show that the set of linear functionals $\{\Phi_{l,m}\}_{l,m\geq 0}$ gives rise to a linear functional Φ on $M^\bullet\otimes Z$. Recall that the module Z is linearly generated by

$$z_1\otimes z_2=\rho_1\Big(D^l_{[i_l]}\Big)\cdots\rho_1\Big(D^1_{[i_1]}\Big)\,\mathbf{1}^{A(V)}\otimes\rho_2\Big(E^m_{[j_m]}\Big)\cdots\rho_2\Big(E^1_{[j_1]}\Big)\,\mathbf{1}^{A(V)}$$

with $D^1, \dots, D^l, E^1, \dots, E^m \in V$, and $i_1, \dots, i_l, j_1, \dots, j_m \in \mathbb{Z}$. For such $z_1 \otimes z_2$ and $w \in M^\bullet$, the linear functional Φ is defined as

$$\Phi\left(w \otimes z_1 \otimes z_2\right) := \left(\frac{1}{2\pi\sqrt{-1}}\right)^{l+m} \oint_{|y_m|=r_{l+m}} \cdots \oint_{|y_1|=r_{l+1}} \oint_{|x_l|=r_l} \cdots \oint_{|x_1|=r_1} \overline{\Phi}_{l,m}(w \otimes D^\bullet \otimes E^\bullet)_{(R_\bullet, S_\bullet)} x_1^{i_1} \cdots x_l^{i_l} y_1^{j_1} \cdots y_m^{j_m} dx_1 \cdots dx_l dy_1 \cdots dy_m$$

where $r_1 > \cdots > r_{l+m} > \max_i |P_i|$ on the rational subset U. This implies that all the P_i's are contained inside the region bounded by the interval of integration, and that the interval of integration avoids the locus $\Delta \cup \Gamma_{P_\bullet} \subset U^{l+m}$. The above defines Φ on the set of generators of $M^\bullet \otimes Z$ and by construction

$$\Phi(w \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)}) = \Phi_{0,0}(w).$$

Step 3(e). We are left to prove that such Φ is compatible with the relations defining Z, and vanishes on $\mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V) \cdot (M^\bullet \otimes_{\mathbb{C}} Z)$.

Recall that Z is spanned by $z_1 \otimes z_2$ as above modulo some relations corresponding to the weak associativity of V as a V-module. Since the linear functionals $\Phi_{l,m}$, and hence Φ, are defined in terms of meromorphic sections arising from the vertex operators $Y^{M^i}(-, z)$, it follows that the relations among elements of Z are preserved, hence Φ is indeed a linear functional on $M^\bullet \otimes Z$.

Finally, we verify that Φ vanishes on $\mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)\,(M^\bullet \otimes_{\mathbb{C}} Z)$. For this, fix σ in $\mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)$. On the affine $C \setminus P_\bullet \sqcup Q_\bullet$, the section σ can be described as the tensor product of a section $\mathcal{B}$ of $\mathcal{V}_C$ and a 1-differential μ. One can choose a rational subset $U \subset C \setminus Q_\bullet$ as in *Step 3(c)* such that $\mathcal{V}_C$ can be trivialized on U. Therefore, the restriction σ_{P_i} of σ from $C \setminus P_\bullet \sqcup Q_\bullet$ to the punctured formal disk $D^\times_{P_i}$ can be described as

$$A \otimes \mu_{P_i}, \qquad \text{for some } A \in V \text{ independent of } P_i,$$

where μ_{P_i} is the Laurent series expansion of μ at P_i. Similarly, after trivializing $\mathcal{V}_C$ on a rational subset containing the points $Q_\bullet$, the restriction σ_{Q_i} of σ to $D^\times_{Q_i}$ can be described as $B \otimes \mu_{Q_i}$, for some $B \in V$ independent of Q_i. For $w \otimes z_1 \otimes z_2 \in M^\bullet \otimes Z$, one has

$$\begin{aligned} (31) \quad & \Phi\left(w \otimes \sigma_{Q_1}(z_1) \otimes z_2\right) + \Phi\left(w \otimes z_1 \otimes \sigma_{Q_2}(z_2)\right) \\ &= \Phi\left(w \otimes \mathrm{Res}_{s_1=0} Y(B, s_1) z_1 \mu_{Q_1} \otimes z_2\right) + \Phi\left(w \otimes z_1 \otimes \mathrm{Res}_{s_2=0} Y(B, s_2) z_2 \mu_{Q_2}\right) \\ &= \mathrm{Res}_{s_1=0} \Phi\left(w \otimes Y(B, s_1) z_1 \otimes z_2\right) \mu_{Q_1} + \mathrm{Res}_{s_2=0} \Phi\left(w \otimes z_1 \otimes Y(B, s_2) z_2\right) \mu_{Q_2}. \end{aligned}$$

From the definition of Φ in terms of the linear functionals $\{\Phi_{l,m}\}_{l,m}$, and the identity $\Phi_{l+1,m} = \Phi_{l,m+1}$ one has that

$$\Phi\left(w \otimes Y(B, s)(z_1) \otimes z_2\right) = \Phi\left(w \otimes z_1 \otimes Y(B, s)(z_2)\right)$$

for some formal variable s. In particular,

$$\Phi\left(w \otimes Y(B, s_1)(z_1) \otimes z_2\right) \mu_{Q_1} \quad \text{and} \quad \Phi\left(w \otimes z_1 \otimes Y(B, s_2)(z_2)\right) \mu_{Q_2}$$

are the Laurent series expansions at Q_1 and Q_2, respectively, of the meromorphic 1-differential on C, with poles at the points $P_\bullet \sqcup Q_\bullet$ and regular elsewhere, given by

$$\Phi\left(w \otimes \mathcal{Y}(\mathcal{B})(z_1) \otimes z_2\right)\, \mu,$$

where $\Phi(w\otimes\mathcal{Y}(\mathcal{B})(z_1)\otimes z_2)$ is the regular function on $C \setminus P_\bullet \sqcup Q_\bullet$ whose fiber over a point T with formal coordinate t is given by $\Phi(w\otimes Y(\mathcal{B}|_T, t)(z_1)\otimes z_2)$. From the residue theorem and the fact that $\mathcal{B}|_{P_i} = A$, it follows that (31) is equal to

$$-\sum_{i=1}^{n} \mathrm{Res}_{t_i=0}\, \Phi\left(w \otimes Y(A, t_i)(z_1) \otimes z_2\right) \mu_{P_i}.$$

By definition

$$\Phi\left(\sigma_{P_\bullet}(w) \otimes z_1 \otimes z_2\right) = \sum_{i=1}^{n} \mathrm{Res}_{t_i=0}\, \Phi\left(Y(A, t_i)(w) \otimes z_1 \otimes z_2\right) \mu_{P_i},$$

where

$$Y(A, t_i)(w) := w_1 \otimes \cdots \otimes Y(A, t_i)(w_i) \otimes \cdots \otimes w_n$$

for $w = w_1 \otimes \cdots \otimes w_n$. Thus we are left to show that

$$\mathrm{Res}_{t_i=0}\, \Phi\left(w \otimes Y(A, t_i)(z_1) \otimes z_2\right) \mu_{P_i} = \mathrm{Res}_{t_i=0}\, \Phi\left(Y(A, t_i)(w) \otimes z_1 \otimes z_2\right) \mu_{P_i}$$

for each i. To verify this, fix $w \otimes A \otimes v \in M^\bullet \otimes V^{\otimes l+m+1}$. The left-hand side is computed by means of $\Phi_{l+1,m}\left(w \otimes A \otimes v\right)$, while the right-hand side by means of $\Phi_{l,m}\left(Y(A, t_i)(w) \otimes v\right)$. Considering the point corresponding to the element A to be in $D_{P_i}^\times$ and fixing the points corresponding to the element $v \in V^{\otimes l+m}$, $\Phi_{l+1,m}\left(w \otimes A \otimes v\right)$ becomes a meromorphic function on D_{P_i}, denoted $\Phi_{l+1,m}\left(w \otimes A \otimes v\right)_{(t_i)}$. The desired identity follows from

$$\Phi_{l+1,m}\left(w \otimes A \otimes v\right)_{(t_i)} = \Phi_{l,m}\left(Y(A, t_i)(w) \otimes v\right)$$

which holds from propagation of vacua (this is similar to [FBZ, §10.3.2]).

The above argument gives

$$\Phi\left(w \otimes \sigma_{Q_1}(z_1) \otimes z_2\right) + \Phi\left(w \otimes z_1 \otimes \sigma_{Q_2}(z_2)\right) = -\Phi\left(\sigma_{P_\bullet}(w) \otimes z_1 \otimes z_2\right).$$

It follows that Φ vanishes on $\sigma\left(M^\bullet \otimes_{\mathbb{C}} Z\right)$ for all $\sigma \in \mathcal{L}_{C\setminus P_\bullet \sqcup Q_\bullet}(V)$, hence Φ lies in the source of $h^\vee$. □

7. Proof of the Factorization Theorem

Here we prove our main result, which we state in complete detail below. For this let us first set some notation. Let $(C, P_\bullet)$ be a stable n-pointed curve with exactly one node, denoted Q. Let $\widetilde{C} \to C$ be the normalization of C, let Q_+ and $Q_- \in \widetilde{C}$ be the two preimages of Q, and set $Q_\bullet = (Q_+, Q_-)$. The curve $\widetilde{C}$ may not be connected. Suppose $M^1, \ldots, M^n$ are V-modules, set $M^\bullet = \otimes_{i=1}^n M^i$, and let $\mathscr{W}$ be the set of all simple V-modules. Consider the map

$$M^\bullet \to \bigoplus_{W\in\mathscr{W}} M^\bullet \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W', \qquad u \mapsto \underset{W\in\mathscr{W}}{\oplus} u \otimes \mathbf{1}^{W_0} \tag{32}$$

where $\mathbf{1}^{W_0} = \mathrm{id}_{W_0} \in \mathrm{End}(W_0) \cong W_0 \otimes W_0^\vee$. Here W_0 is the degree zero space of the module $W = \oplus_{i\geq 0} W_i$. Recall that, by definition, the vector spaces W_0 and $W_0^\vee$ are finite-dimensional.

Theorem 7.0.1. *Let $V = \oplus_{i\geq 0} V_i$ be a rational, C_2-cofinite vertex operator algebra with $V_0 \cong \mathbb{C}$. The map* (32) *gives rise to a canonical isomorphism of vector spaces*

$$\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)} \cong \bigoplus_{W\in\mathscr{W}} \mathbb{V}\left(V; M^\bullet \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W'\right)_{(\widetilde{C}, P_\bullet \sqcup Q_\bullet)}.$$

The proof we give here roughly follows the outline of the proof in [NT, §8.6], with the generalizations to coinvariants defined using the chiral Lie algebra instead of Zhu's Lie algebra, and for curves of arbitrary genus, made possible by Propositions 3.3.1 and 6.2.1.

Proof. By definition (21), due to propagation of vacua, we can reduce to the case $C \setminus P_\bullet$ affine, after possibly adding more marked points P_i and corresponding modules V. Fix formal coordinates t_i at P_i, for each $i = 1, \ldots, n$, and $s_\pm$ at $Q_\pm$, so that we have Lie algebra homomorphisms

$$\mathcal{L}_{C\setminus P_\bullet}(V) \to \mathfrak{L}_{P_\bullet}(V) \quad \text{and} \quad \mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V) \to \mathfrak{L}_{P_\bullet}(V) \oplus \mathfrak{L}_{Q_\bullet}(V).$$

In the following, we show that (32) induces a canonical isomorphism

$$M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V)} \cong \bigoplus_{W\in\mathscr{W}} \left(M^\bullet \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W'\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)}. \tag{33}$$

Being independent of the choice of formal coordinates t_i and $s_\pm$, this implies the assertion made in the Factorization Theorem.

We will argue that there is a commutative diagram

$$\begin{array}{ccc} M^\bullet_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\})} & \xrightarrow[\cong]{h} & (M^\bullet \otimes_{\mathbb{C}} Z)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)} \\ \big\downarrow & & \big\downarrow \\ M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V)} & \xrightarrow[\cong]{f} & \left(M^\bullet \otimes_{\mathbb{C}} \overline{Z}\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)}. \end{array}$$

Then after Lemma 6.1.1, the isomorphism f gives (33).

Step 1. The top horizontal isomorphism h is given by Proposition 6.2.1.

Step 2. We argue that there is an inclusion

$$\iota \colon \mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\}) \hookrightarrow \mathcal{L}_{C\setminus P_\bullet}(V).$$

Indeed, by Proposition 3.3.1 an element of $\mathcal{L}_{C\setminus P_\bullet}(V)$ can be realized as σ in $\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)$ such that $\sigma_{Q_\pm} \in \mathfrak{L}(V)_{\leq 0}$ and the restrictions $\left[\sigma_{Q_\pm}\right]_0$ of $\sigma_{Q_\pm}$ to $\mathfrak{L}(V)_0$ satisfy $\left[\sigma_{Q_-}\right]_0 = \vartheta\left[\sigma_{Q_+}\right]_0$. In particular, $\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\})$ from (25) is a Lie subalgebra of $\mathcal{L}_{C\setminus P_\bullet}(V)$, since its elements satisfy $\sigma_{Q_\pm} \in \mathfrak{L}(V)_{<0}$.

Step 3. To show that the bottom horizontal map f is an isomorphism, it remains to verify that the kernel of the two vertical maps coincide.

Step 3(a). The kernel of the left vertical map is the space

$$K = \mathcal{L}_{C\setminus P_\bullet}(V)\left(M^\bullet_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V,\{Q_+,Q_-\})}\right).$$

Note that for σ in $\mathcal{L}_{C\setminus P_\bullet}(V)/\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\})$, the formula for the bracket (3) gives

$$\left[\varphi(\sigma), \varphi\left(\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\})\right)\right] \subset \varphi\left(\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\})\right),$$

where φ is as in (16). It follows that σ acts on the source of h. The left vertical map is thus the quotient by the action of $\mathcal{L}_{C\setminus P_\bullet}(V)/\mathcal{L}_{\widetilde{C}\setminus P_\bullet}(V, \{Q_+, Q_-\})$.

Step 3(b). We conclude the argument by showing that the right vertical map coincides with the quotient by $h(K)$. Recall that $h(w) = w \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)}$, for $w \in M^\bullet$. For σ in $\mathcal{L}_{C\setminus P_\bullet}(V)$, one has

$$\begin{aligned}\sigma_{P_\bullet}(w) \otimes \mathbf{1}^{A(V)} \otimes \mathbf{1}^{A(V)} \equiv & -w \otimes \sigma_{Q_+}\left(\mathbf{1}^{A(V)}\right) \otimes \mathbf{1}^{A(V)} \\ & - w \otimes \mathbf{1}^{A(V)} \otimes \sigma_{Q_-}\left(\mathbf{1}^{A(V)}\right) \quad \mathrm{mod}\ \mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)\left(M^\bullet \otimes Z\right).\end{aligned}$$

Since the elements of $\mathscr{U}(V)_{<0}$ act trivially on $A(V)$ this is congruent to

$$-w \otimes \left[\sigma_{Q_+}\right]_0\left(\mathbf{1}^{A(V)}\right) \otimes \mathbf{1}^{A(V)} - w \otimes \mathbf{1}^{A(V)} \otimes \vartheta\left[\sigma_{Q_+}\right]_0\left(\mathbf{1}^{A(V)}\right)$$

modulo $\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)\left(M^\bullet \otimes Z\right)$. From lemma 6.1.1, one has

$$\begin{aligned}\left(M^\bullet \otimes Z\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)} & \cong \bigoplus_{W,Y\in\mathscr{W}} \frac{M^\bullet \otimes W \otimes W_0^\vee \otimes Y \otimes Y_0^\vee}{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)\left(M^\bullet \otimes W \otimes W_0^\vee \otimes Y \otimes Y_0^\vee\right)} \\ & \cong \bigoplus_{W,Y\in\mathscr{W}} \left(M^\bullet \otimes W \otimes Y\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)} \otimes W_0^\vee \otimes Y_0^\vee.\end{aligned}$$

Recall Lemma 1.8.1 on the action of $\mathfrak{L}(V)$ on $W_0^\vee$. It follows that the image of K in $\left(M^\bullet \otimes Z\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)}$ is

$$\bigoplus_{W,Y\in\mathscr{W}} \left(M^\bullet \otimes W \otimes Y\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)} \otimes \mathscr{I}\left(W_0^\vee, Y_0^\vee\right)$$

with $\mathscr{I}\left(W_0^\vee, Y_0^\vee\right) \subset W_0^\vee \otimes Y_0^\vee$ linearly spanned by

$$\psi_W \circ \vartheta\left(A_{[k-1]}\right) \otimes \psi_Y + \psi_W \otimes \psi_Y \circ A_{[k-1]},$$

where $\psi_W \in W_0^\vee$, $\psi_Y \in Y_0^\vee$ for $W, Y \in \mathscr{W}$, and $A \in V_k$, for $k \geq 0$. One has

$$W_0^\vee \otimes Y_0^\vee / \mathscr{I}\left(W_0^\vee, Y_0^\vee\right) = \mathrm{Hom}_{A(V)}\left(W_0, Y_0^\vee\right),$$

and by Schur's Lemma, this is isomorphic to $\mathbb{C}$ when $Y = W'$ and zero otherwise. This and the description of $\overline{Z}$ from Lemma 6.1.1 imply that, after taking the quotient of $(M^\bullet \otimes_{\mathbb{C}} Z)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)}$ by the kernel of the left vertical map, one obtains $\left(M^\bullet \otimes_{\mathbb{C}} \overline{Z}\right)_{\mathcal{L}_{\widetilde{C}\setminus P_\bullet \sqcup Q_\bullet}(V)}$, hence the statement. □

8. Sewing and local freeness

In this section we prove VB Corollary . For this, we start with Theorem 8.3.1, a refined version of the Factorization Theorem. This requires the notion of formal smoothings, reviewed below.

8.1. **Formal smoothings.** For a $\mathbb{C}$-algebra R with smooth $\mathrm{Spec}(R)$, let $\mathscr{C}_0 \to S_0 = \mathrm{Spec}(R)$ be a flat family of stable n-pointed curves with a single node defined by a section Q and with the n smooth points given by sections $P_\bullet = (P_1, \dots, P_n)$. Assume that $\mathscr{C}_0 \setminus P_\bullet(S_0)$ is affine over S_0. Up to an étale base change of S_0 of degree two, we can normalize $\mathscr{C}_0$ and obtain a smooth family of $(n+2)$-pointed curves $\widetilde{\mathscr{C}_0} \to S_0$ with sections $P_\bullet \sqcup (Q_+, Q_-)$, where $Q_\pm(S_0) \in \widetilde{\mathscr{C}_0}$ are the preimages of the node in $\mathscr{C}_0/S_0$. Fix formal coordinates s_+ and s_- at $Q_+(S_0)$ and $Q_-(S_0)$, respectively. Such coordinates determine a *smoothing* of $(\mathscr{C}_0, P_\bullet)$ over $S = \mathrm{Spec}(R[\![q]\!])$. That is, a flat family $\mathscr{C} \to S = \mathrm{Spec}(R[\![q]\!])$ with sections $P_\bullet = (P_1, \dots, P_n)$ such that the general fiber is smooth and the special fiber is identified with $\mathscr{C}_0 \to S_0$. The family $\widetilde{\mathscr{C}_0} \to S_0$ extends to a family of smooth curves $\widetilde{\mathscr{C}} \to S = \mathrm{Spec}(R[\![q]\!])$, with $n+2$ sections $P_\bullet$, Q_+, and Q_-, with the special fiber identified with $\widetilde{\mathscr{C}_0} \to S_0$, and which fits in the diagram

$$\begin{array}{ccc} \widetilde{\mathscr{C}} & \longrightarrow & \mathscr{C} \\ & \searrow \quad \swarrow & \\ {\scriptstyle P_\bullet, Q_+, Q_-} & S = \mathrm{Spec}(R[\![q]\!]) & {\scriptstyle P_\bullet} \end{array}.$$

The formal coordinate at $Q_\pm(S_0)$ extends to a formal coordinate, still denoted $s_\pm$, at $Q_\pm(S)$ — that is, $s_\pm$ is generator of the ideal of the completed local $R[\![q]\!]$-algebra of $\widetilde{\mathscr{C}}$ at $Q_\pm(R)$ — such that $s_+ s_- = q$. For more details, see [Loo, p. 457] and [ACG, pp. 184-5]. We emphasize that the existence of such families holds over the formal base $S = \mathrm{Spec}(R[\![q]\!])$, or equivalently, over the complex open unit disk around S_0 in the analytic category, but fails over a more general base. Moreover, one still has that $\widetilde{\mathscr{C}} \setminus P_\bullet(S)$ and $\mathscr{C} \setminus P_\bullet(S)$ are affine over S.

8.2. **The sheaf of coinvariants.** The construction of the chiral Lie algebra in §3 and of coinvariants in §4 can be extended over an arbitrary smooth base. One thus obtains a sheaf of Lie algebras $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ and a sheaf of coinvariants $\left(\otimes_{i=1}^n M^i \otimes \mathscr{O}_S\right)_{\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)}$, for given V-modules $M^1, \dots, M^n$ and a choice of formal coordinates t_i at the points P_i. One can remove the assumption that $\mathscr{C} \setminus P_\bullet(S)$ is affine over S using propagation of vacua as in §4, and furthermore obtain a *sheaf of coinvariants* $\mathbb{V}(V;M^\bullet)_{(\mathscr{C}/S,P_\bullet)}$ on S independent of the formal coordinates t_i via a descent along a torsor as in [FBZ, §17], [DGT1].

Similarly to Proposition 5.1.1, we obtain the following result.

Theorem 8.2.1. *Let $V=\oplus_{i\geq 0}V_i$ be a C_2-cofinite vertex operator algebra with $V_0\cong\mathbb{C}$. For any collection of finitely generated V-modules $M^1,\dots,M^n$, the sheaf of coinvariants $\mathbb{V}(V;M^\bullet)_{(\mathscr{C}/S,P_\bullet)}$ is a coherent $\mathscr{O}_S$-module.*

Proof. The argument runs similarly to the one for Proposition 5.1.1, with the elements of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ interpreted as elements of $\mathcal{L}_{\widetilde{\mathscr{C}}\setminus P_\bullet\sqcup Q_\bullet}(V)$ satisfying appropriate gluing given by the smoothing construction. □

Moreover, the Factorization Theorem holds for such families $\mathscr{C}_0\to S_0$ as follows:

Theorem 8.2.2. *Let $V=\oplus_{i\geq 0}V_i$ be a rational, C_2-cofinite vertex operator algebra with $V_0\cong\mathbb{C}$. The map* (32) *induces a canonical $\mathscr{O}_{S_0}$-module isomorphism*

$$\mathbb{V}(V;M^\bullet)_{(\mathscr{C}_0/S_0,P_\bullet)}\cong\bigoplus_{W\in\mathcal{W}}\mathbb{V}\left(V;M^\bullet\otimes W\otimes W'\right)_{\left(\widetilde{\mathscr{C}}_0/S_0,P_\bullet\sqcup Q_\bullet\right)}.$$

8.3. **Sewing.** Given a simple V-module $W=\oplus_{i\geq 0}W_i$, define

$$\mathbf{1}^W:=\sum_{i\geq 0}\mathbf{1}^{W_i}q^i\in W\otimes W'\otimes\mathbb{C}[\![q]\!],$$

where $\mathbf{1}^{W_i}:=\mathrm{id}_{W_i}\in\mathrm{End}(W_i)\cong W_i\otimes W_i^\vee$. Consider the map

$$M^\bullet\longrightarrow M^\bullet\otimes W\otimes W'\otimes\mathbb{C}[\![q]\!],\qquad u\mapsto\oplus_{W\in\mathscr{W}}u\otimes\mathbf{1}^W. \tag{34}$$

The following result extends [NT, Theorem 8.4.6].

Sewing Theorem 8.3.1. *Let $V=\oplus_{i\geq 0}V_i$ be a rational, C_2-cofinite vertex operator algebra with $V_0\cong\mathbb{C}$, and set $M^\bullet=\bigotimes_{i=1}^n M^i$ for V-modules M^i. The map* (34) *induces a canonical $\mathscr{O}_{S_0}[\![q]\!]$-module isomorphism Ψ such that the following diagram commutes*

$$\begin{array}{ccc}\mathbb{V}(V;M^\bullet)_{(\mathscr{C}/S,P_\bullet)} & \xrightarrow{\Psi} & \bigoplus_{W\in\mathscr{W}}\mathbb{V}\left(V;M^\bullet\otimes W\otimes W'\right)_{\left(\widetilde{\mathscr{C}}_0/S_0,P_\bullet\sqcup Q_\bullet\right)}\otimes_{\mathscr{O}_{S_0}}\mathscr{O}_{S_0}[\![q]\!]\\ \big\downarrow & & \big\downarrow\\ \mathbb{V}(V;M^\bullet)_{(\mathscr{C}_0/S_0,P_\bullet)} & \xrightarrow{\cong} & \bigoplus_{W\in\mathscr{W}}\mathbb{V}\left(V;M^\bullet\otimes W\otimes W'\right)_{\left(\widetilde{\mathscr{C}}_0/S_0,P_\bullet\sqcup Q_\bullet\right)}.\end{array}$$

Remark 8.3.2. Observe that by Theorems 8.2.2 and 8.3.1, there is a canonical isomorphism

$$\mathbb{V}(V;M^\bullet)_{(\mathscr{C}/S,P_\bullet)}\cong\mathbb{V}\left(V;M^\bullet\right)_{(\mathscr{C}_0/S_0,P_\bullet)}[\![q]\!].$$

In particular, this means that to the non-trivial deformation $\mathscr{C}$ of $\mathscr{C}_0$, there corresponds a trivial deformation of the space of conformal blocks.

Proof of Theorem 8.3.1. As in the proof of the Factorization Theorem, we can reduce to the case $\mathscr{C} \setminus P_\bullet$ affine over S, we fix formal coordinates t_i at $P_i(S)$, for $i = 1, \ldots, n$, and $s_\pm$ at $Q_\pm(S)$, and show that (34) induces a canonical $R[\![q]\!]$-module isomorphism, still denoted Ψ, such that the following diagram commutes

$$\begin{array}{ccc} \left(M^\bullet \otimes \mathscr{O}_S\right)_{\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)} & \stackrel{\Psi}{\to} & \bigoplus_{W\in\mathscr{W}} \left(M^\bullet \otimes W \otimes W' \otimes \mathscr{O}_{S_0}\right)_{\mathcal{L}_{\widetilde{\mathscr{C}}_0\setminus P_\bullet\sqcup Q_\bullet}(V)} \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{S_0}[\![q]\!] \\ \downarrow & & \downarrow \\ \left(M^\bullet \otimes \mathscr{O}_{S_0}\right)_{\mathcal{L}_{\mathscr{C}_0\setminus P_\bullet}(V)} & \xrightarrow{\cong} & \bigoplus_{W\in\mathscr{W}} \left(M^\bullet \otimes W \otimes W' \otimes \mathscr{O}_{S_0}\right)_{\mathcal{L}_{\widetilde{\mathscr{C}}_0\setminus P_\bullet\sqcup Q_\bullet}(V)}. \end{array}$$

The construction will be independent of the choice of the formal coordinates t_i and $s_\pm$, hence this will imply the statement.

Step 1. We start by showing that (34) induces a well-defined map Ψ between spaces of coinvariants. For this, it is enough to show that for each $\sigma \in \mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ and $W \in \mathscr{W}$, one has $\sigma_{P_\bullet}(M^\bullet) \otimes \mathbf{1}^W = \sigma\left(M^\bullet \otimes \mathbf{1}^W\right)$, or equivalently

$$\left(\sigma_{Q_+} \otimes 1 + 1 \otimes \sigma_{Q_-}\right)\left(\mathbf{1}^W\right) = 0. \tag{35}$$

This vanishing follows by using relative stable differentials on $\mathscr{C}/S$ to describe elements of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$, in the same spirit of the proof of Proposition 3.3.1. Namely, an element of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ can be lifted to one in

$$\oplus_{k\geq 0} H^0\left(\mathscr{C} \setminus P_\bullet, \left(\omega_{\mathscr{C}/S}^{\otimes 1-k}\right)^{\oplus \dim V_k}\right).$$

Members of this set can in turn be described as elements in

$$\oplus_{k\geq 0} H^0\left(\widetilde{\mathscr{C}} \setminus P_\bullet \sqcup Q_\bullet, \left(\omega_{\widetilde{\mathscr{C}}/S}^{\otimes 1-k}\right)^{\oplus \dim V_k}\right)$$

satisfying certain order and residue conditions. Before expressing these conditions, let us first note that a section μ of $\omega_{\widetilde{\mathscr{C}}/S}^{\otimes 1-k}$ on $\widetilde{\mathscr{C}} \setminus P_\bullet$ satisfies

$$\mu_{Q_+} = \sum_{i,j\geq 0} a_{i+k-1,j}\, s_+^{i-j+k-1} q^j \,(ds_+)^{1-k},$$
$$\mu_{Q_-} = \sum_{i,j\geq 0} b_{i,j+k-1}\, s_-^{j-i+k-1} q^i \,(ds_-)^{1-k}$$

with

$$a_{i+k-1,j} = (-1)^{k-1} b_{i,j+k-1}, \qquad \text{for } i,j,k \geq 0.$$

This description follows from the identities $s_+ ds_- + s_- ds_+ = 0$, $s_+ s_- = q$, and the order conditions on relative stable differentials prescribing that the above two sums are only over nonnegative values of i and j (see e.g., [NT, §8.5]). The order conditions for relative stable differentials imply analogous order conditions for elements of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$. Moreover, the gluing conditions for elements of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ are induced from the gluing conditions for sections of $\mathcal{V}_C$, see §2.5. Namely, by linearity we can reduce to the case $\sigma_{Q_+} = A \otimes \mu$ in $V_k \otimes \omega_{\widetilde{\mathscr{C}}/S}^{\otimes 1-k} \cong \left(\omega_{\widetilde{\mathscr{C}}/S}^{\otimes 1-k}\right)^{\oplus \dim V_k}$ for $k \geq 0$. It follows that σ_{Q_+} and σ_{Q_-} act as

$$\begin{aligned}\sigma_{Q_+} &= \sum_{i,j\geq 0} a_{i+k-1,j}\, A_{[i-j+k-1]} q^j,\\ \sigma_{Q_-} &= \sum_{i,j\geq 0} b_{i,j+k-1} \sum_{l\geq 0} \frac{1}{l!}\left(L_1^l A\right)_{[j-i+k-1-l]} q^i\\ &= \sum_{i,j\geq 0} a_{i+k-1,j}\, \vartheta\left(A_{[i-j+k-1]}\right) q^i.\end{aligned}$$

This is as in the end of the proof of Proposition 3.3.1: the isomorphism between $V \otimes s_\pm^{-1} ds_\pm \cong \oplus_{k\geq 0} V_k \otimes s_\pm^{k-1}(ds_\pm)^{1-k}$ and the stalk of $\mathcal{L}_{\widetilde{\mathscr{C}}\setminus P_\bullet \sqcup Q_\bullet}(V)$ at $Q_\pm$ maps $L_1^l A \otimes s_\pm^{-1} ds_\pm$ to $A_{[k-l-1]}$, for $A \in V_k$. Hence, the vanishing (35) follows from the identity

$$\left(A_{[i-j+k-1]} \otimes 1 + 1 \otimes \vartheta\left(A_{[i-j+k-1]}\right) q^{i-j}\right) \mathbf{1}^W = 0 \tag{36}$$

established in [NT, Lemma 8.7.1] (note that there is a sign of difference between the involution ϑ used in this paper and the involution used in [NT]). Thus we conclude that the map Ψ is well-defined and makes the diagram above commute.

Step 2. Since (i) the target of Ψ is a free $\mathscr{O}_{S_0}[\![q]\!]$-module of finite rank, (ii) the source is finitely generated (Theorem 8.2.1), and (iii) Φ is an isomorphism modulo q (Theorem 8.2.2), Nakayama's lemma implies that Ψ is an isomorphism (this is as in [TUY], [Loo], [NT]). □

Remark 8.3.3. By [DGT1, §7] there is a twisted logarithmic $\mathcal{D}$-module structure on sheaves of coinvariants. Specifically, both $\mathbb{V}(V;M^\bullet)_{(\mathscr{C}/S,P_\bullet)}$ and $\bigoplus_{W\in\mathscr{W}} \mathbb{V}\left(M^\bullet \otimes W \otimes W'\right)_{\widetilde{\mathscr{C}}_0\setminus P_\bullet \sqcup Q_\bullet} \underset{\mathscr{O}_{S_0}}{\otimes} \mathscr{O}_{S_0}[\![q]\!]$ are equipped with a projective action by those derivations of $\mathscr{O}_{S_0}[\![q]\!]$ preserving the ideal generated by q.

From this point of view, one can interpret Theorem 8.3.1 as a spectral decomposition for the action of $D = q\partial_q$. Observe that $\mathscr{T}_S(\log S_0) = \mathscr{O}_S \otimes_{\mathscr{O}_{S_0}} \mathscr{T}_{S_0} + \mathscr{O}_S q\partial_q$, and by $\mathscr{O}_{S_0}[\![q]\!]$-linearity, the action of $\mathscr{T}_{S_0}$ commutes with Ψ since it acts on both spaces by coefficient-wise derivation.

We claim that for every $u \in M^\bullet$ and $W \in \mathscr{W}$, one has

$$\Psi \circ D = (q\partial_q + c_W \mathrm{id}) \circ \Psi, \tag{37}$$

where c_W is the conformal dimension of W: $L_0(w) = (c_W + \deg w)w$, for homogeneous $w \in W$. As in [Loo, Lemma 33], the smoothing construction gives a natural way to lift $q\partial_q$ to a derivation of $(\mathscr{O}_{S_0}[\![s_+]\!] \oplus \mathscr{O}_{S_0}[\![s_-]\!])[\![q]\!]$ of the form

$$D_{\underline{a}} := (s_+\partial_{s_+}, 0) + \sum_{m,n\geq 0} a_{m,n}(s_+^{m-n+1} q^n \partial_{s_+}, -s_-^{n-m+1} q^m \partial_{s_-})$$

for some $a_{m,n} \in \mathscr{O}_{S_0}$ (there is a typo in [Loo, Lemma 33], nevertheless the computation there is compatible with the above statement). As a special case of the identity (36), we observe that

$$(L_{m-n}q^n, -L_{n-m}q^m)\mathbf{1}^W = 0.$$

It follows that

$$\begin{aligned} q\partial_q(\Psi(u)) = D_{\underline{a}}(u \otimes \mathbf{1}^W) &= D_{\underline{a}}(u) \otimes \mathbf{1}^W + u \otimes D_{\underline{a}}(\mathbf{1}^W) \\ &= D(u) \otimes \mathbf{1}^W + u \otimes (-L_0, 0)(\mathbf{1}^W) \\ &= D(u) \otimes \mathbf{1}^W - c_W u \otimes \mathbf{1}^W \\ &= \Psi(D(u) - c_W(u)), \end{aligned}$$

hence the claim.

8.4. **Proof of VB Corollary.** By means of Theorems 8.2.1 and 8.3.1, one concludes that the sheaf $\mathbb{V}(V; M^\bullet)$ is a vector bundle of finite rank on $\overline{\mathcal{M}}_{g,n}$, as in [TUY], [Sor, §2.7], [Loo], [NT]. Below we sketch the argument for completeness.

The sheaf $\mathbb{V}(V; M^\bullet)$ on $\mathcal{M}_{g,n}$ has finite-dimensional fibers (Proposition 5.1.1) and is equipped with a projectively flat connection [DGT1]. As in [TUY], see also [Sor, §2.7], it follows that $\mathbb{V}(V; M^\bullet)$ is coherent and locally free of finite rank on $\mathcal{M}_{g,n}$. After Theorem 8.2.1 and gluing the sheaf as in [BL2], it follows that the sheaf $\mathbb{V}(V; M^\bullet)$ is also coherent on $\overline{\mathcal{M}}_{g,n}$. It remains to show that $\mathbb{V}(V; M^\bullet)$ is locally free on $\overline{\mathcal{M}}_{g,n}$. For this, consider a family of nodal curves $(\mathscr{C}_0 \to \operatorname{Spec}(R), P_\bullet)$, and for simplicity assume that it has only one node. Consider its formal smoothing $(\mathscr{C} \to \operatorname{Spec}(R[\![q]\!]), P_\bullet)$ as described in §8.1. Theorem 8.3.1 implies that $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet)}$ is locally free of finite rank, hence we conclude the argument. For families of curves with more nodes, one proceeds similarly by induction on the number of nodes. □

9. Examples

In the Factorization Theorem, VB Corollary and in the results leading up to them, including the Sewing Theorem 8.3.1, we assume that the vertex operator algebras we consider satisfy three hypotheses: (1) $V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} V_i$ with $V_0 \cong \mathbb{C}$; (2) V is rational; and (3) V is C_2-cofinite. It is worth to mention that under condition (1), it has been proved that (2) and (3) are equivalent to V being *regular*, that is, any weak V-module is a direct sum of irreducible ordinary V-modules [DLM2, Rmk 3.2], [Li3], [ABD, Thm 4.5]. Here we briefly discuss examples where these hypotheses are known to hold.

9.1. **Virasoro vertex algebras.** Recall the Virasoro Lie algebra Vir described in §1.1.1, and consider its Lie subalgebra $\mathrm{Vir}_{\geq 0} := \mathbb{C}K \oplus z\mathbb{C}[\![z]\!]\partial_z$. Given complex numbers c and h, the Verma module for Vir of central charge c and highest weight h is $M_{c,h} := U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_{\geq 0})} \mathbb{C}\mathbf{1}$, where $\mathbb{C}\mathbf{1}$ is given the structure of a $\mathrm{Vir}_{\geq 0}$-module by setting $L_{p>0}\mathbf{1} = 0$, $L_0\mathbf{1} = h\mathbf{1}$, and $K\mathbf{1} = c\mathbf{1}$.

There is a unique maximal proper submodule $J_{c,h} \subset M_{c,h}$. For $h = 0$, $J_{c,0}$ contains a submodule generated by the singular vector $L_{-1}\mathbf{1} \in M_{c,0}$ [FF]. Set

$$L_{c,h} := M_{c,h}/J_{c,h}, \qquad M_c := M_{c,0}/\langle L_{-1}\mathbf{1}\rangle, \qquad \text{and} \qquad Vir_c := L_{c,0}.$$

If $c \neq c_{p,q} := 1 - \frac{6(p-q)^2}{pq}$, with relatively prime $p, q \in \mathbb{N}$ such that $1 < p < q$, then $M_c \cong Vir_c$, that is, $J_{c,0} = \langle L_{-1}\mathbf{1}\rangle$, while for $c = c_{p,q}$, the submodule $J_{c,0}$ is generated by two singular vectors [FF]. By [FZ, Thm 4.3], M_c and Vir_c are conformal vertex algebras and satisfy property (1). By [FZ, Thm 4.6], the Zhu algebra satisfies $A(M_c) \cong \mathbb{C}[x]$. As $\mathbb{C}[x]$ has infinitely many 1-dimensional irreducible representations, M_c is never rational, and Vir_c is not rational for $c \neq c_{p,q}$. On the other hand, for $c = c_{p,q}$, Vir_c is rational [Wan, Thm 4.2]: the Zhu algebra is $A(Vir_c) \cong \mathbb{C}[x]/\langle G_{p,q}(x)\rangle$, for some $G_{p,q}(x) \in \mathbb{C}[x]$, and the irreducible representations of Vir_c are given by the modules $L_{c,h}$, for $h = \frac{(np-mq)^2-(p-q)^2}{4pq}$, with $0 < m < p$ and $0 < n < q$. Moreover, Vir_c is C_2-cofinite for $c = c_{p,q}$ [DLM4, Lemma 12.3] (see also [Ara1, Prop. 3.4.1]).

9.2. **Irreducible affine vertex algebras from simple Lie algebras.** We briefly describe the irreducible conformal vertex algebra $L_\ell(\mathfrak{g})$ associated to a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ and a level $\ell \in \mathbb{Z}_{>0}$ [FZ, §2], [LL, §6.2] (see also [NT, §A.1.1]).

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra, and θ the longest of some choice of positive roots. Normalize the Cartan-Killing form $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ so that $(\theta, \theta) = 2$. Let $\widehat{\mathfrak{g}} := \mathbb{C}K \oplus \mathfrak{g} \otimes \mathbb{C}(\!(t)\!)$ be the corresponding affine Lie algebra with K central and bracket

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (A, B)\,(\mathrm{Res}_{t=0}\, f dg)\, K.$$

For W a $\mathfrak{g}$-module, we consider the Verma $\widehat{\mathfrak{g}}$-module $W_\ell = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} W$, where W is given the structure of a module over $\widehat{\mathfrak{g}}_{\geq 0} := \mathbb{C}K \oplus \mathfrak{g} \otimes \mathbb{C}[\![t]\!]$ by letting $\mathfrak{g} \otimes t\mathbb{C}[\![t]\!]$ act by zero and K by $\ell \cdot \mathrm{id}_W$. For instance, let W_λ be the irreducible $\mathfrak{g}$-module associated to a weight $\lambda \in \mathfrak{h}^*$ and write $W_{\lambda,\ell} = (W_\lambda)_\ell$. Let $J_{\lambda,\ell} \subset W_{\lambda,\ell}$ be the unique maximal proper submodule, and let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. By [FZ, Thm 2.4.1], for $\ell \neq -h^\vee$, $W_{0,\ell}$ and the quotient $L_\ell(\mathfrak{g}) := W_{0,\ell}/J_{0,\ell}$ are conformal vertex algebras satisfying hypothesis (1), and $L_\ell(\mathfrak{g})$ is irreducible. Frenkel and Zhu show in [FZ, Thm 3.1.3] that $L_\ell(\mathfrak{g})$ is rational: the Zhu algebra $A(L_\ell(\mathfrak{g})) = U(\mathfrak{g})/\langle e_\theta^{\ell+1}\rangle$ is a quotient of the universal enveloping algebra for $\mathfrak{g}$ (here e_θ is a nonzero element of the one-dimensional root space $\mathfrak{g}_\theta$), which is semisimple with irreducible $U(\mathfrak{g})/\langle e_\theta^{\ell+1}\rangle$-modules equal to the finite-dimensional irreducible $\mathfrak{g}$-modules with highest weight λ such that $(\lambda, \theta) \leq \ell$, a finite set [Kac]. Consequently, the irreducible $L_\ell(\mathfrak{g})$-modules are $L_{\lambda,\ell} := W_{\lambda,\ell}/J_{\lambda,\ell}$, for $\lambda \in \mathfrak{h}^*$, $(\lambda, \theta) \leq \ell$. That $L_\ell(\mathfrak{g})$ is C_2-cofinite follows from [Zhu2] (see also [DLM4, Prop. 12.6], [Ara1, Proposition 3.5.1]). We emphasize that Verma modules associated to simple $A(L_\ell(\mathfrak{g}))$-modules are simple. This follows from two facts: (a)

$L_\ell(\mathfrak{g})$-modules are integrable $\widehat{\mathfrak{g}}$-modules from the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [TK]; and (b) the complete reducibility for integrable $\widehat{\mathfrak{g}}$-modules from the category $\mathcal{O}$ [Kac, Thm 10.7]. Note how complete reducibility can fail outside the BGG category $\mathcal{O}$ for modules of affine Lie algebras. Indeed we know that Verma modules of affine Lie algebras are indecomposable but not irreducible. On the other hand, modules for irreducible affine vertex algebras coincides with modules from the category $\mathcal{O}$, hence Verma modules of irreducible affine vertex algebras are irreducible.

9.3. **The moonshine module $V^\natural$.** A rational vertex operator algebra V is called *holomorphic* if V is the unique irreducible V-module. If properties (1) and (3) also hold, then V must have central charge divisible by 8 [DM1]. One example of such a conformal vertex algebra is the moonshine module $V^\natural$, of central charge 24, relevant for a number of reasons, including the fact that $\mathrm{Aut}(V^\natural)$ is the monster sporadic group [FLM1].

9.4. **Even lattice vertex algebras.** Let L be a positive-definite even lattice of finite rank, that is, a free abelian group of finite rank, together with a positive definite bilinear form $(\cdot,\cdot)$ for which $(\alpha,\alpha)\in 2\mathbb{Z}$ for any $\alpha\in L$. Set $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}} L$ (which inherits the bilinear form from L). Consider the Heisenberg Lie algebra $\widehat{\mathfrak{h}}$ associated to $\mathfrak{h}$ and $(\cdot,\cdot)$, defined as the one-dimensional central extension of $\mathfrak{h}(\!(t)\!)$:

$$0\to\mathbb{C}\mathbf{1}\to\widehat{\mathfrak{h}}\to\mathfrak{h}(\!(t)\!)\to 0,$$

with bracket given by $[A\otimes f(t), B\otimes g(t)] = -(A,B)\mathrm{Res} f(t)g'(t)dt\cdot\mathbf{1}$. Let $\widetilde{\mathcal{H}}$ be the quotient of the completion of the universal enveloping algebra $U(\widehat{\mathfrak{h}})$ by the ideal generated by the element $\mathbf{1}-1$ (here $\mathbf{1}$ is the central element of $\widehat{\mathfrak{h}}$ and 1 is the unit of the associative algebra $U(\widehat{\mathfrak{h}})$). The algebra $\widetilde{\mathcal{H}}$ has topological generators h_n, for $h\in\mathfrak{h}$ and $n\in\mathbb{Z}$. Given $\lambda\in\mathfrak{h}$ we consider the (infinitely many) Fock representations of $\widetilde{\mathcal{H}}$

$$\pi_\lambda=\widetilde{\mathcal{H}}\otimes_{\widetilde{\mathcal{H}}_{\geq 0}}\mathbb{C}v_\lambda,$$

where $\widetilde{\mathcal{H}}_{\geq 0}$ has topological generators h_n where $h\in\mathfrak{h}$ and $n\in\mathbb{N}$. To define the structure of an $\widetilde{\mathcal{H}}_{\geq 0}$-module on the one-dimensional vector space $\mathbb{C}v_\lambda$, for $h\in\mathfrak{h}$, one sets $h_nv_\lambda=0$ for $n>0$, and $h_0v_\lambda=(\lambda,h)v_\lambda$. There is a vertex operator algebra structure on $V_L=\bigoplus_{\lambda\in L}\pi_\lambda$, and the lattice vertex algebra V_L satisfies property (1) [Bor], [FLM1]. As a vector space, one has a canonical isomorphism $V_L\cong\pi_0\otimes\mathbb{C}[L]$. Here $\mathbb{C}[L]$ is the group algebra associated to L (elements are e^λ, for $\lambda\in L\setminus\{0\}$). By [Don], V_L is rational, and the complete set of V_L-modules is given by $\{V_{L+\lambda}:\lambda\in L'/L\}$, where $L'=\{\lambda\in L\otimes_{\mathbb{Z}}\mathbb{Q}:(\lambda,\mu)\in\mathbb{Z},\forall\mu\in L\}$ is the dual lattice. In [DLM1, Thm 3.4] the Zhu algebra $A(V_L)$ is described explicitly and in [DLM1, Prop. 2.10] the simple $A(V_L)$-modules are constructed, giving an alternative proof of rationality. By [DLM4] the V_L are C_2-cofinite.

9.5. **Exceptional W-algebras.** Arakawa [Ara3] shows that a large number of simple W-algebras, including all the minimal series principal W-algebras [FKW] and the exceptional W-algebras of Kac-Wakimoto [KW], are C_2-cofinite. He proves that the minimal series principal W-algebras and a large subclass of exceptional affine W-algebras are rational [Ara2], [Ara4], [AvE] and satisfy property (1).

9.6. **Orbifold vertex algebras.** Given a vertex algebra V and a group of automorphisms G of V, the orbifold vertex algebra V^G is the G-invariant vertex subalgebra of V. Conjecturally, if V is C_2-cofinite (and rational) and G is finite, then V^G will also be C_2-cofinite (and rational). Both of these are known to be true in case G is a finite and solvable group, when V is a simple vertex operator algebra satisfying condition (1) [Miy, CM]. In this case, V and V^G satisfy (1), (2), and (3).

9.7. **Commutants.** To a vertex algebra V and a vertex subalgebra $\mathcal{A} \subset V$, Frenkel and Zhu [FZ] defined a vertex algebra $Com(\mathcal{A}, V)$ arising as a vertex subalgebra that commutes with $\mathcal{A}$. Conjecturally, if $\mathcal{A}$ and V are both C_2-cofinite and rational, then $Com(\mathcal{A}, V)$ will be as well. This was proved for the case of parafermion vertex operator algebras [DR], after prior work [ALY, DLWY, DLY, DW1, DW2, DW3]. These vertex operator algebras satisfy (1) [DR]. Parafermions have also been constructed as orbifold algebras.

9.8. **Tensor products.** Let $V^1, \dots, V^m$ be vertex algebras. Then $V = \bigotimes_{i=1}^m V^i$ is a vertex algebra and by [DMZ], V is rational if and only if V^i is rational for all i. Moreover, the tensor product of regular conformal vertex algebras is regular [DLM2].

Appendix A. Zhu's Lie algebra and isomorphic coinvariants

For a smooth curve C and a quasi-primary generated vertex algebra V with $V_0 \cong \mathbb{C}$, in addition to the chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ (§3.1), one also has Zhu's Lie algebra $\mathfrak{g}_{C\setminus P_\bullet}(V)$, reviewed in §A.1.

In Proposition A.2.1, we show that when defined, $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is isomorphic to the image of $\mathcal{L}_{C\setminus P_\bullet}(V)$ under the Lie algebra homomorphism $\varphi_{\mathcal{L}}$ (Proposition A.2.1). Nagatomo and Tsuchiya extend the definition of $\mathfrak{g}_{C\setminus P_\bullet}(V)$ to stable pointed rational curves [NT], and they indicate that their coinvariants are equivalent to those studied by Beilinson and Drinfeld in [BD], suggesting they knew that Proposition A.2.1 holds in that case.

A quasi-primary vector is an element $A \in V$ such that $L_1 A = 0$, and V is quasi-primary generated if and only if $L_1 V_1 = 0$ [DLM5]. A vertex algebra $V = \bigoplus_{i\geq 0} V_i$ with $V_0 \cong \mathbb{C}$ satisfies $L_1 V_1 = 0$ if and only if $V \cong V'$ (see [FHL, §5.3] and [DM2, §2]). In particular, in the results of Huang [Hua3] and Codogni [Cod], the vertex algebras studied are quasi-primary generated.

A.1. **The Lie algebra $\mathfrak{g}_{C\setminus P_\bullet}(V)$.** In [Zhu2], given a smooth pointed curve $(C, P_\bullet)$ and a quasi-primary generated vertex operator algebra V for which $V_0 \cong \mathbb{C}$, Zhu defines a Lie algebra $\mathfrak{g}_{C\setminus P_\bullet}(V)$, generalizing the construction of Tsuchiya, Ueno, and Yamada for affine Lie algebras. Namely, consider

$$\mathfrak{g}_{C\setminus P_\bullet}(V) := \varphi_{\mathfrak{g}}\left(\oplus_{k\geq 0} V_k \otimes H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right)\right) \tag{38}$$

where

$$\varphi_{\mathfrak{g}}\colon\ \oplus_{k\geq 0} V_k \otimes H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) \to \oplus_{i=1}^n \mathfrak{L}_{t_i}(V) \tag{39}$$

is the map induced by

$$B \otimes \mu \mapsto \left(\mathrm{Res}_{t_i=0}\, Y[B, t_i]\mu_{P_i}(dt_i)^k\right)_{i=1,\dots,n}.$$

Here t_i is a formal coordinate at the point P_i, $Y[B, t_i] := \sum_{k\in\mathbb{Z}} B_{[k]} t_i^{-k-1}$, and μ_{P_i} is the Laurent series expansion of μ at P_i, the image of μ via

$$H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) \to H^0\left(D^\times_{P_i}, \omega_C^{\otimes 1-k}\right) \simeq_{t_i} \mathbb{C}(\!(t_i)\!)(dt_i)^{1-k}.$$

When V is assumed to be quasi-primary generated with $V_0 \cong \mathbb{C}$, Zhu shows that $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is a Lie subalgebra of $\mathfrak{L}(V)^{\oplus n}$. The argument uses that any *fixed* smooth algebraic curve admits an atlas such that all transition functions are Möbius transformations. Transition functions between charts on families of curves of arbitrary genus are more general, hence the need to consider the more involved construction for the chiral Lie algebra based on the $(\mathrm{Aut}\,\mathcal{O})$-twist of V in §2.

A.2. **Isomorphism of coinvariants.** When $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is well-defined and $C \setminus P_\bullet$ is affine, one can define the space of coinvariants $M^\bullet_{\mathfrak{g}_{C\setminus P_\bullet}(V)}$ as the quotient of the $\mathfrak{L}(V)^{\oplus n}$-module $M^\bullet$ by the action of the Lie subalgebra $\mathfrak{g}_{C\setminus P_\bullet}(V)$ of $\mathfrak{L}(V)^{\oplus n}$. These spaces were introduced in [Zhu1], and studied also in [AN, NT]. Recall the homomorphisms $\varphi_{\mathcal{L}}$ from (16) and $\varphi_{\mathfrak{g}}$ from (39).

Proposition A.2.1. *When $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is well-defined (§A.1), one has*

$$\mathrm{Im}(\varphi_{\mathcal{L}}) \cong \mathrm{Im}(\varphi_{\mathfrak{g}}).$$

It follows that there exists an isomorphism of vector spaces

$$M^\bullet_{\mathfrak{g}_{C\setminus P_\bullet}(V)} \;\cong\; M^\bullet_{\mathcal{L}_{C\setminus P_\bullet}(V)}.$$

Proof. One has

$$\begin{aligned}\oplus_{k\geq 0} V_k \otimes H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) &\cong H^0\left(C \setminus P_\bullet, \oplus_{k\geq 0} V_k \otimes \omega_C^{\otimes 1-k}\right)\\ &\cong H^0\left(C \setminus P_\bullet, \oplus_{k\geq 0}\left(\omega_C^{\otimes 1-k}\right)^{\oplus \dim V_k}\right).\end{aligned} \tag{40}$$

From Lemma 2.6.1, one has $\mathrm{gr}_\bullet \mathcal{V}_C \cong \oplus_{k\geq 0}\left(\omega_C^{\otimes -k}\right)^{\oplus \dim V_k}$. It follows that

$$H^0\left(C \setminus P_\bullet, \oplus_{k\geq 0}\left(\omega_C^{\otimes 1-k}\right)^{\oplus \dim V_k}\right) \cong H^0(C \setminus P_\bullet, \mathrm{gr}_\bullet \mathcal{V}_C \otimes \omega_C).$$

Now by Lemma 2.6.2,

$$H^0\left(C\setminus P_\bullet, \mathrm{gr}_\bullet\mathcal{V}_C\otimes\omega_C\right)\cong H^0\left(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C\right). \tag{41}$$

On the other hand, as $C\setminus P_\bullet$ is assumed to be affine, one has

$$\mathcal{L}_{C\setminus P_\bullet}(V)\cong H^0\left(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C\right)/\nabla H^0\left(C\setminus P_\bullet,\mathcal{V}_C\right).$$

The map $\varphi_{\mathcal{L}}$ is induced from the composition

$$H^0\left(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C\right)\to\oplus_{i=1}^n H^0\left(D_{P_i}^\times,\mathcal{V}_C\otimes\omega_C\right)\to\oplus_{i=1}^n\mathfrak{L}_{t_i}(V). \tag{42}$$

The first map is canonical and obtained by restricting sections; the second is (14). By [FBZ, §6.6.9], sections in $\nabla H^0\left(D_{P_i}^\times,\mathcal{V}_C\right)$ act trivially. Hence (42) induces a map from the Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ to $\oplus_{i=1}^n\mathfrak{L}_{t_i}(V)$. It follows that the image of $\varphi_{\mathcal{L}}$ coincides with the image of $H^0\left(C\setminus P_\bullet,\mathcal{V}_C\otimes\omega_C\right)$ in $\oplus_{i=1}^n\mathfrak{L}_{t_i}(V)$ via (42). Composing (40) and (41), and by the definition of $\varphi_{\mathfrak{g}}$ in (39), the image of the map in (42) coincides with the image of $\varphi_{\mathfrak{g}}$. □

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