VERTEX ALGEBRAS OF CohFT-TYPE

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Dedicated to Bill Fulton on the occasion of his 80th birthday.

Abstract. Representations of certain vertex algebras, here called of CohFT-type, can be used to construct vector bundles of coinvariants and conformal blocks on moduli spaces of stable curves [DGT2]. We show that such bundles define semisimple cohomological field theories. As an application, we give an expression for their total Chern character in terms of the fusion rules, following the approach and computation in [MOP+2] for bundles given by integrable modules over affine Lie algebras. It follows that the Chern classes are tautological. Examples and open problems are discussed.

Vertex algebras, fundamental in a number of areas of mathematics and mathematical physics, have recently been shown to be a source of new constructions for vector bundles on moduli of curves [FBZ, DGT2]. In particular, given an n-tuple of modules M^i over a vertex algebra V satisfying certain natural hypotheses (stated in §2.1), one may construct the *vector bundle of coinvariants* $\mathbb{V}_g(V; M^\bullet)$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves of genus g [DGT2]. The fiber at a pointed curve $(C, P_\bullet)$ is the vector space of coinvariants, i.e., the largest quotient of $\otimes_{i=1}^n M^i$ by the action of a Lie algebra determined by $(C, P_\bullet)$ and the vertex algebra V.

Such vector bundles generalize the classical coinvariants of integrable modules over affine Lie algebras [TK, TUY]. Bundles of coinvariants from vertex algebras have much in common with their classical counterparts. For instance, both support a projectively flat logarithmic connection [TUY, DGT1] and satisfy factorization [TUY, DGT2], a property that makes recursive arguments about ranks and Chern classes possible.

Following [MOP+2], bundles of coinvariants from integrable modules over affine Lie algebras give cohomological field theories (CohFTs for short). Here we show the same is true for their generalizations. We say that a vertex algebra V is of *CohFT-type* if V satisfies the hypotheses of §2.1. We prove:

Theorem 1. *For a vertex algebra V of CohFT-type, the collection consisting of the Chern characters of all vector bundles of coinvariants from finitely-generated V-modules forms a semisimple cohomological field theory.*

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In particular, the ranks of the bundles of coinvariants form a *topological quantum field theory* (TQFT), namely, the degree zero part of the CohFT. As such, the ranks are recursively determined by the *fusion rules*, that is, the dimension of spaces of coinvariants on a three-pointed rational curve (Proposition 3.2.1). The fusion rules have been computed in the literature for many classes of vertex algebras of CohFT-type (see §5 for a few examples).

In fact, the CohFTs from Theorem 1 are determined by the fusion rules. Indeed, after work of Givental and Teleman [Giv1, Giv2, Tel], a semisimple CohFT is determined by its TQFT part together with some additional structure (see also [Pan]). As in [MOP$^+$2], the explicit computation of the Atiyah algebra giving rise to the projectively flat logarithmic connection allows one to determine the recursion. As the Atiyah algebra in the case of bundles of coinvariants from vertex algebras was determined in [DGT1], one is able to extend the reconstruction of the CohFTs of coinvariants from affine Lie algebras in [MOP$^+$2] to the general case of vertex algebras.

Namely, following [MOP$^+$2], there exists a polynomial $P_V(a_\bullet)$ with coefficients in $H^*\big(\overline{\mathcal{M}}_{g,n}\big)$, explicitly given in §4, such that the following holds:

Corollary 1. *For a vertex algebra V of CohFT-type and an n-tuple $M^\bullet$ of simple V-modules with M^i of conformal dimension a_i, the Chern character of the vector bundle of coinvariants $\mathbb{V}_g(V;M^\bullet)$ is*

$$\mathrm{ch}\left(\mathbb{V}_g(V;M^\bullet)\right) = P_V(a_\bullet) \quad \text{in } H^*\big(\overline{\mathcal{M}}_{g,n}\big).$$

By Corollary 1, Chern classes of bundles of coinvariants defined by vertex algebras of CohFT-type lie in the tautological ring of $\overline{\mathcal{M}}_{g,n}$. As an explicit example of the classes, the first Chern class in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ is given by:

Corollary 2. *Let V be a vertex algebra of CohFT-type and central charge c, and let M^i be simple V-modules of conformal dimension a_i. Then*

$$c_1\left(\mathbb{V}_g(V;M^\bullet)\right) = \mathrm{rank}\,\mathbb{V}_g(V;M^\bullet)\left(\frac{c}{2}\lambda + \sum_{i=1}^n a_i\psi_i\right) - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i,I} b_{i:I}\delta_{i:I},$$

$$\text{with } \; b_{\mathrm{irr}} = \sum_{W\in\mathscr{W}} a_W \cdot \mathrm{rank}\,\mathbb{V}_{g-1}\left(V;M^\bullet\otimes W\otimes W'\right)$$

$$\text{and } \; b_{i:I} = \sum_{W\in\mathscr{W}} a_W \cdot \mathrm{rank}\,\mathbb{V}_i\left(V;M^I\otimes W\right)\cdot \mathrm{rank}\,\mathbb{V}_{g-i}\left(V;M^{I^c}\otimes W'\right).$$

Here $\mathscr{W}$ is the set of finitely many simple V-modules; a_W is the conformal dimension of a simple V-module W (§1.3); for $I\subseteq[n]=\{1,\dots,n\}$, we set $M^I := \otimes_{i\in I}M^i$; and the last sum is over i,I such that $i\in\{0,\dots,g\}$ and $I\subseteq[n]$, modulo the relation $(i,I)\equiv(g-i,I^c)$.

As for the first Chern class in Corollary 2, the Chern classes depend on the central charge of the vertex algebra and the conformal dimensions (or weights) of the modules. Since V is of CohFT-type, the central charge and the conformal dimensions of the modules are rational [DLM2].

Plan of paper: We start in §1 with some background on vertex algebras. In particular, there we describe the sheaf of coinvariants $\mathbb{V}_g(V; M^\bullet)$ and its dual, the sheaf of conformal blocks. In §2 we review a selection of results on vector bundles defined by representations of vertex algebras of CohFT-type, mainly from [DGT1] and [DGT2], which will be used to prove the statements above. Theorem 1 is proved in §3 and Corollary 1 in §4. In §5 we review the invariants necessary to compute the Chern classes in several examples, including the moonshine module vertex algebra $V^\natural$ and even lattice vertex algebras. We discuss the problem for commutant and orbifold vertex algebras, illustrating with parafermion vertex algebras, and orbifolds of lattice and parafermion vertex algebras.

From this and prior work, it is clear that the vector bundle of coinvariants from modules over vertex algebras of CohFT-type have a number of properties in common with their classical analogues, for which much has already been discovered. For instance, bundles of coinvariants defined from modules over affine Lie algebras are particularly interesting on $\overline{\mathcal{M}}_{0,n}$, where they are globally generated, and their sections define morphisms [Fak]. In particular, by studying their Chern classes one can learn about the maps they define. In §6 we discuss questions one might explore with this in mind.

1. Background

Here we briefly review vertex algebras, their modules, related Lie algebras, and the vector spaces of coinvariants they define. We refer the reader to [FHL, LL, FBZ, DGT1, DGT2] for details. We use notation as in [DGT2], where further information and references on these topics can be found.

1.1. **The Virasoro algebra.** The *Witt (Lie) algebra* $\operatorname{Der}\mathcal{K}$ is the Lie algebra $\mathbb{C}(\!(z)\!)\partial_z$ generated by $L_p := -z^{p+1}\partial_z$, for $p \in \mathbb{Z}$, with Lie bracket given by $[L_p, L_q] = (p-q)L_{p+q}$.

The *Virasoro (Lie) algebra* Vir is a central extension of $\operatorname{Der}\mathcal{K}$ which is generated by a formal vector K and the elements L_p, for $p \in \mathbb{Z}$, with Lie bracket given by

$$[K, L_p] = 0, \qquad [L_p, L_q] = (p-q)L_{p+q} + \frac{1}{12}(p^3 - p)\delta_{p+q,0}\, K.$$

A representation of Vir has *central charge* $c \in \mathbb{C}$ if $K \in \mathrm{Vir}$ acts as $c \cdot \mathrm{id}$.

1.2. **Vertex operator algebras.** A *vertex operator algebra* is a four-tuple $\big(V, \mathbf{1}^V, \omega, Y(\cdot, z)\big)$ with: $V = \oplus_{i\geq 0} V_i$ a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-vector space with $\dim V_i < \infty$; two distinguished elements $\mathbf{1}^V \in V_0$ (the *vacuum vector*) and $\omega \in V_2$ (the *conformal vector*); a linear map $Y(\cdot, z)\colon V \to \mathrm{End}(V)\,[\![z, z^{-1}]\!]$ that assigns to $A \in V$ the *vertex operator* $Y(A, z) := \sum_{i\in\mathbb{Z}} A_{(i)} z^{-i-1}$. These data are required to satisfy suitable axioms, see e.g., [DGT2, §1.1]. We review below some of the consequences which will be used in what follows. When no confusion arises, we refer to the four-tuple as V.

The Fourier coefficients of the fields $Y(\cdot, z)$ endow V with a series of products indexed by $\mathbb{Z}$, that is, $A *_i B := A_{(i)}B$, for $A, B \in V$. These products are *weakly* commutative and *weakly* associative.

The *conformal structure* of V realizes the Fourier coefficients $\omega_{(i)}$ as a representation of the Virasoro algebra on V via the identifications $L_p = \omega_{(p+1)}$ and $K = c \cdot \mathrm{id}_V$ for a constant $c \in \mathbb{C}$ called the *central charge* of V. Moreover, L_0 is required to act as a degree operator on V, i.e., $L_0|_{V_i} = i \cdot \mathrm{id}_{V_i}$, and L_{-1} (the *translation operator*) is given by $L_{-1}A = A_{(-2)}\mathbf{1}^V$, for $A \in V$.

As a consequence of the axioms, one has $A_{(i)}V_k \subseteq V_{k+\deg(A)-i-1}$ for homogeneous $A \in V$ [Zhu], hence the *degree* of the operator $A_{(i)}$ is defined as $\deg A_{(i)} := \deg(A) - i - 1$.

1.3. **Modules of vertex operator algebras.** There are a number of ways to define a module over a vertex operator algebra V. We take a V-module M to be a module over the universal enveloping algebra $\mathscr{U}(V)$ of V (defined by I. Frenkel and Zhu [FZ], see also [FBZ, §5.1.5]) satisfying three finiteness properties. Namely, we assume that: (i) M is a finitely generated $\mathscr{U}(V)$-module; (ii) $F^0 \mathscr{U}(V)v$ is finite-dimensional, for every v in M; and (iii) for every v in M, there exists a positive integer k such that $F^k \mathscr{U}(V)v = 0$. These conditions are as in [NT, Def. 2.3.1]. Here, $F^k \mathscr{U}(V) \subset \mathscr{U}(V)$ is the vector subspace topologically generated by compositions of operators with total degree less than or equal to $-k$.

E. Frenkel and Ben-Zvi [FBZ, Thm 5.1.6] showed that there is an equivalence of categories between $\mathscr{U}(V)$-modules satisfying property (iii) and the so-called *weak* V-modules, which a priori are not graded. However, with the additional assumptions (i) and (ii), one can show the modules have a grading by the natural numbers. Such a V-module consists of a pair $\left(M, Y^M(\cdot, z)\right)$, where $M = \oplus_{i\geq 0} M_i$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-vector space, and $Y^M(\cdot, z) \colon V \to \mathrm{End}(M)\left[\!\left[z, z^{-1}\right]\!\right]$ is a linear function that assigns to $A \in V$ an $\mathrm{End}(M)$-valued vertex operator $Y^M(A, z) := \sum_{i\in\mathbb{Z}} A^M_{(i)} z^{-i-1}$. Moreover, by condition (i), if $A \in V$ is homogeneous, then $A^M_{(i)} M_k \subseteq M_{k+\deg(A)-i-1}$.

The V-modules we work with are also known in the literature as finitely generated *admissible* V-modules (see for instance, [ABD] for the definitions of weak and admissible V-modules).

As for V, one has that M is also naturally equipped with an action of the Virasoro algebra with central charge c, induced by the identification of $\omega^M_{(p+1)}$ with L_p. When M is a simple V-module, there exists $a_M \in \mathbb{C}$, called the *conformal dimension* (or *conformal weight*) of M, such that $L_0(v) = (\deg(v) + a_M)v$, for every homogeneous v in M [Zhu].

The vertex algebra V is a module over itself, sometimes referred to as the *adjoint module* [LL, §4.1] or the *trivial module*. In what follows the set of simple modules over V is denoted $\mathscr{W}$.

1.4. **Contragredient modules.** Contragredient modules provide a notion of duality for V-modules. We recall their definition following [FHL, §5.2].

For a vertex algebra V and a V-module $\left(M = \oplus_{i\geq 0} M_i, Y^M(-,z)\right)$, its *contragredient module* is $\left(M', Y^{M'}(-,z)\right)$, where M' is the graded dual of M, that is, $M' := \oplus_{i\geq 0} M_i^\vee$, with $M_i^\vee := \mathrm{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, and $Y^{M'}(-,z)\colon V \to \mathrm{End}\,(M')\,[\![z,z^{-1}]\!]$ is the unique linear map determined by

$$\left\langle Y^{M'}(A,z)\psi, m\right\rangle = \left\langle \psi, Y^M\left(e^{zL_1}(-z^{-2})^{L_0}A, z^{-1}\right)m\right\rangle$$

for $A\in V$, $\psi\in M'$, and $m\in M$. Here $\langle\cdot,\cdot\rangle$ is the natural pairing between a vector space and its graded dual.

1.5. **The Lie algebra ancillary to V.** The *Lie algebra ancillary to V* is defined as the quotient

$$\mathfrak{L}(V) = \mathfrak{L}_t(V) := \left(V\otimes \mathbb{C}(\!(t)\!)\right)\big/\mathrm{Im}\,\partial,$$

where t is a formal variable and $\partial := L_{-1}\otimes \mathrm{id}_{\mathbb{C}(\!(t)\!)} + \mathrm{id}_V\otimes \partial_t$. The image of $A\otimes t^i \in V\otimes \mathbb{C}(\!(t)\!)$ in $\mathfrak{L}(V)$ is denoted by $A_{[i]}$. Observe that $\mathfrak{L}(V)$ is spanned by series of the form $\sum_{i\geq i_0} c_i\, A_{[i]}$, for $A\in V$, $c_i\in\mathbb{C}$, and $i_0\in\mathbb{Z}$. The Lie bracket is induced by

$$\left[A_{[i]}, B_{[j]}\right] := \sum_{k\geq 0}\binom{i}{k}\left(A_{(k)}\cdot B\right)_{[i+j-k]}.$$

There is a canonical Lie algebra isomorphism between $\mathfrak{L}(V)$ and the current Lie algebra in [NT]. In what follows, the formal variable t is interpreted as a formal coordinate at a point P on an algebraic curve. A coordinate-free description of $\mathfrak{L}(V)$ is provided in §1.6.

For a V-module M, the Lie algebra homomorphism $\mathfrak{L}(V)\to\mathrm{End}(M)$ defined by

$$\sum_{i\geq i_0} c_i A_{[i]} \mapsto \mathrm{Res}_{z=0}\, Y^M(A,z)\sum_{i\geq i_0} c_i z^i dz$$

induces an action of $\mathfrak{L}(V)$ on M. For instance, $A_{[i]}$ acts as the Fourier coefficient $A_{(i)}$ of the vertex operator $Y^M(A,z)$.

1.6. **The vertex algebra bundle and the chiral Lie algebra.** Let $(C,P_\bullet)$ be a stable n-pointed curve. As illustrated in [FBZ] for smooth curves and in [DGT2] for stable curves, one can construct a vector bundle $\mathscr{V}_C$ (the *vertex algebra bundle*) on C whose fiber at each point of C is (non-canonically) isomorphic to V. For a smooth open subset $U\subset C$ admitting a global coordinate (e.g., if there exists an étale map $U\to\mathbb{A}^1$), the choice of a global coordinate on U gives a trivialization $\mathscr{V}_C|_U\cong V\times U$. The vertex algebra bundle $\mathscr{V}_C$ is constructed via descent along the torsor of formal coordinates at points in C. The bundle $\mathscr{V}_C$ is naturally equipped with a flat connection $\nabla\colon\mathscr{V}_C\to\mathscr{V}_C\otimes\omega_C$ such that, up to the choice of a formal coordinate t_i at P_i, one can identify

$$H^0\left(D^\times_{P_i}, \mathscr{V}_C\otimes\omega_C/\mathrm{Im}\nabla\right)\cong\mathfrak{L}_{t_i}(V). \tag{1}$$

Here $D_{P_i}^{\times}$ is the punctured formal disk about the marked point $P_i \in C$. As shown in [FBZ, §§19.4.14, 6.6.9], the isomorphism (1) induces the structure of a Lie algebra independent of coordinates on the left-hand side.

The *chiral Lie algebra* is defined as

$$\mathscr{L}_{C\setminus P_\bullet}(V) := H^0\left(C \setminus P_\bullet, \mathscr{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right).$$

This space has indeed the structure of a Lie algebra after [FBZ, §19.4.14].

1.7. **The action of the chiral Lie algebra on V-modules.** Consider the linear map φ given by restriction of sections from $C \setminus P_\bullet$ to the n punctured formal disks $D_{P_i}^{\times}$ using the formal coordinates t_i at P_i:

$$\varphi\colon \mathscr{L}_{C\setminus P_\bullet}(V) \to \oplus_{i=1}^n \mathfrak{L}_{t_i}(V).$$

After [FBZ, §19.4.14], φ is a homomorphism of Lie algebras. The map φ thus induces an action of $\mathscr{L}_{C\setminus P_\bullet}(V)$ on $\mathfrak{L}(V)^{\oplus n}$-modules which is used to construct coinvariants.

1.8. **Sheaves of coinvariants and conformal blocks.** We briefly recall how to construct sheaves of coinvariants on $\overline{\mathcal{M}}_{g,n}$, and refer to [DGT1, §5] for a detailed exposition. To a stable n-pointed curve $(C, P_\bullet)$ of genus g such that $C \setminus P_\bullet$ is affine, and to V-modules $M^1, \dots, M^n$, we associate the space of coinvariants

$$\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)} := M^\bullet_{\mathscr{L}_{C\setminus P_\bullet}(V)} = M^\bullet / \mathscr{L}_{C\setminus P_\bullet}(V) \cdot M^\bullet,$$

where $M^\bullet = \otimes_{i=1}^n M^i$. Thanks to the *propagation of vacua*, it is possible to define these spaces also when $C \setminus P_\bullet$ is not affine via a direct limit. Carrying out the construction relatively over $\overline{\mathcal{M}}_{g,n}$, one defines the quasi-coherent *sheaf of coinvariants* $\mathbb{V}_g(V; M^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ assigned to $M^\bullet$. The dual sheaf $\mathbb{V}_g(V; M^\bullet)^\dagger$ is the *sheaf of conformal blocks* assigned to $M^\bullet$.

A brief history of coinvariants and conformal blocks and of the work on their properties can be found in [DGT2, §§0.1 and 0.2].

2. Vector bundles of coinvariants

Here we review a number of results about vector bundles of coinvariants defined from representations of vertex algebras satisfying certain natural hypotheses. Motivated by the new results proved here, we name vertex algebras satisfying such hypotheses as *vertex algebras of CohFT-type*.

2.1. **Vertex algebras of CohFT-type.** We define a vertex algebra V to be *of CohFT-type* if V is a vertex operator algebra such that:

(I) $V = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} V_i$ with $V_0 \cong \mathbb{C}$;

(II) V is *rational*, that is, every finitely generated V-module is a direct sum of simple V-modules; and

(III) V is C_2-*cofinite*, that is, the subspace

$$C_2(V) := \mathrm{span}_{\mathbb{C}} \left\{ A_{(-2)}B \,:\, A, B \in V \right\}$$

has finite codimension in V.

The set $\mathscr{W}$ of simple modules over a rational vertex algebra is finite, and a simple module $M = \oplus_{i\geq 0} M_i$ over a rational vertex algebra satisfies $\dim M_i < \infty$ [DLM1].

The assumptions (I)-(III) on the vertex algebra have been found in [DGT2] to imply that the sheaves of coinvariants are in fact vector bundles:

Theorem 2.1.1 ([DGT2, VB Corollary]). *For a vertex algebra V of CohFT-type, the sheaf of coinvariants $\mathbb{V}_g(V; M^\bullet)$ assigned to finitely generated admissible V-modules $M^1, \ldots, M^n$ is a vector bundle of finite rank on the moduli space $\overline{\mathcal{M}}_{g,n}$.*

Since V is rational and C_2-cofinite, the central charge of V and the conformal dimension of every simple V-module are rational numbers [DLM2]. When V is of CohFT-type, the Chern character of the bundles of coinvariants form a cohomological field theory, as we verify below, hence the name.

2.2. **The connection.** Following [DGT1], the vector bundles $\mathbb{V}_g(V; M^\bullet)$ support a projectively flat logarithmic connection. We can explicitly describe this using the language of Atiyah algebras [BS]. Given a line bundle L on a variety, the Atiyah algebra $\mathcal{A}_L$ is the sheaf of first order differential operators acting on L. An analoguous construction holds for a virtual line bundle L^x, where $x \in \mathbb{C}$, yielding the Atiyah algebra $x\mathcal{A}_L$ [Tsu]. With this terminology, the connection on $\mathbb{V}_g(V; M^\bullet)$ is explicitly described as follows:

Theorem 2.2.1 ([DGT1]). *For n simple modules M^i of conformal dimension a_i over a vertex algebra V of CohFT-type and central charge c, the Atiyah algebra $\frac{c}{2}\mathcal{A}_\Lambda + \sum_{i=1}^n a_i \mathcal{A}_{\Psi_i}$ with logarithmic singularities along the boundary Δ acts on $\mathbb{V}_g(V; M^\bullet)$, specifying a twisted logarithmic $\mathcal{D}$-module structure.*

Here Λ is the Hodge line bundle on $\overline{\mathcal{M}}_{g,n}$; Ψ_i is the cotangent line bundle at the i-th marked point on $\overline{\mathcal{M}}_{g,n}$; and $\Delta = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is the locus of singular curves. Theorem 2.2.1 generalizes the analogous statement for bundles of coinvariants of integrable representations at a fixed level over affine Lie algebras [Tsu]. This is proved more generally for *quasi-coherent* sheaves of coinvariants in [DGT1, §7].

2.3. **Chern classes on $\mathcal{M}_{g,n}$.** The explicit description of the connection determines the Chern character of the restriction of $\mathbb{V}_g(V; M^\bullet)$ on $\mathcal{M}_{g,n}$:

Corollary 2.3.1 ([DGT1, Corollary 2]). *Let V be a vertex algebra of CohFT-type and central charge c, and let $M^1, \dots, M^n$ be n simple V-modules of conformal dimension a_i. Then*

$$\mathrm{ch}\left(\mathbb{V}_g(V; M^\bullet)\right) = \mathrm{rank}\, \mathbb{V}_g(V; M^\bullet) \cdot \exp\left(\frac{c}{2}\lambda + \sum_{i=1}^n a_i \psi_i\right) \in H^*(\overline{\mathcal{M}}_{g,n}).$$

Here $\lambda = c_1(\Lambda)$ and $\psi_i = c_1(\Psi_i)$. The corollary follows from: (a) a vector bundle E over a smooth base S with an action of the Atiyah algebra $\mathcal{A}_{L^{\otimes a}}$, for $L \to S$ a line bundle and $a \in \mathbb{Q}$, satisfies $c_1(E) = (\mathrm{rank}\, E)\, aL$ [MOP1, Lemma 5]; and (b) the projectively flat connection implies that $\mathrm{ch}(E) = (\mathrm{rank}\, E) \exp(c_1(E)/\mathrm{rank}\, E)$ [Kob, (2.3.3)].

From Corollary 2.3.1, the total Chern class is

$$c\left(\mathbb{V}_g(V; M^\bullet)\right) = \left(1 + \frac{c}{2}\lambda + \sum_{i=1}^n a_i \psi_i\right)^{\mathrm{rank}\, \mathbb{V}_g(V; M^\bullet)} \in H^*(\overline{\mathcal{M}}_{g,n}).$$

2.4. **The factorization property.** After [DGT2, Factorization Theorem], for V of CohFT-type, the bundles $\mathbb{V}_g(V; M^\bullet)$ satisfy the *factorization property.* Assume that the curve C has one nodal point Q, let $\widetilde{C}$ be the partial normalization of C at Q, and let $Q_\bullet = (Q_+, Q_-)$ be the pair of preimages of Q in $\widetilde{C}$.

Theorem 2.4.1 ([DGT2, Factorization Theorem]). *Let V be a vertex algebra of CohFT-type. Then*

$$\mathbb{V}\left(V; M^\bullet\right)_{(C, P_\bullet)} \cong \bigoplus_{W \in \mathscr{W}} \mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{\left(\widetilde{C}, P_\bullet \sqcup Q_\bullet\right)}.$$

When $\widetilde{C} = C_+ \sqcup C_-$ is disconnected, with $Q_\pm \in C_\pm$, one has:

$$\mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{(\widetilde{C}, P_\bullet \sqcup Q_\bullet)} \cong \mathbb{V}\left(V; M_+^\bullet \otimes W\right)_{X_+} \otimes \mathbb{V}\left(V; M_-^\bullet \otimes W'\right)_{X_-}$$

where $X_\pm = (C_\pm, P_\bullet|_{C_\pm} \sqcup Q_\pm)$, and $M_\pm^\bullet$ are the modules at the $P_\bullet$ on $C_\pm$. The factorization property extends in families of nodal curves. More generally, the *sewing property* holds, extending the factorization property over the formal neighborhood of families of nodal curves [DGT2].

2.5. **Finding ranks through recursions.** As a consequence of the factorization property, the rank of $\mathbb{V}_g(V; M^\bullet)$, equal to the dimension of the vector space of coinvariants $\mathbb{V}(V; M^\bullet)_{(C, P_\bullet)}$ at any pointed curve $(C, P_\bullet)$, can be computed when $(C, P_\bullet)$ is maximally degenerate. Using propagation of vacua [FBZ, §10.3.1] and inserting points with the adjoint module V if necessary, one may reduce to the case when all irreducible components of C are rational curves with three special points, and thus the rank may be expressed as sum of products of dimensions of vector spaces of coinvariants on three-pointed rational curves.

As the following two examples show, formulas for the ranks can be readily identified in some simple cases. A third such recursive calculation is carried out for bundles defined from modules over even lattice vertex algebras in Example 5.2.5.

Example 2.5.1. The rank of $\mathbb{V}_1(V;V)$ on $\overline{\mathcal{M}}_{1,1}$ equals the cardinality of the set of simple V-modules $\mathscr{W}$. This follows from factorization and the equality (2) in the next section.

Example 2.5.2. Let V be a vertex algebra of CohFT-type with no nontrivial modules. Bundles of coinvariants of modules over V on $\overline{\mathcal{M}}_{g,n}$ have rank 1.

In §3.2 we use the formalism of semisimple TQFTs to reconstruct the ranks from the fusion rules. Rank computations for bundles defined from modules over even lattice vertex algebras are done from this perspective in Examples 5.2.4 and 5.2.6.

3. Proof of Theorem 1 and TQFT computations

In this section, using the results from §2, we prove Theorem 1, showing bundles of coinvariants defined by representations of vertex algebras of CohFT-type give rise to cohomological field theories (for more on CohFTs, see e.g., [MOP$^+$2, §2]). Theorem 1 is proved as in the case of coinvariants from affine Lie algebras treated in [MOP$^+$2, §3], and we follow their approach.

3.1. **The CohFT of Chern characters of coinvariants.** Let us start by defining the data of the CohFT. Let V be a vertex algebra of CohFT-type. As V is rational, the set $\mathscr{W}$ of simple V-modules is finite. The Hilbert space of the CohFT is $\mathcal{F}(V) := \mathbb{Q}^{\mathscr{W}} = \oplus_{W\in\mathscr{W}}\mathbb{Q}\, h_W$. The CohFT is defined by the classes

$$\mathrm{ch}\left(\mathbb{V}_g(V;M^\bullet)\right) \in H^*\!\left(\overline{\mathcal{M}}_{g,n}\right),$$

for $2g-2+n>0$ and V-modules $M^1,\ldots,M^n$ viewed as elements of $\mathcal{F}(V)$ (hence necessarily finitely-generated), and extending by linearity. The vector space has a pairing η defined by $\eta(h_M,h_N)=\delta_{M,N'}$, where N' is the contragredient of N. In addition, $\mathcal{F}(V)$ has a commutative, associative product $*$ defined by

$$h_M * h_N := \sum_{W\in\mathscr{W}} \dim \mathbb{V}_0\left(V;M,N,W'\right) h_W,$$

for $M,N\in\mathscr{W}$ and extending by linearity, with unit h_V corresponding to the adjoint module V. This product extends linearly to the *fusion algebra* $\mathcal{F}(V)_{\mathbb{C}} := \mathcal{F}(V)\otimes_{\mathbb{Q}}\mathbb{C}$. This is a commutative, associative Frobenius algebra with unit. In particular, one has $\eta(a*b,c)=\eta(a,b*c)$, for $a,b,c\in\mathcal{F}(V)_{\mathbb{C}}$.

Proof of Theorem 1. The axioms necessary to form a CohFT are verified thanks to the factorization property in families [DGT2, Thm 8.2.2], propagation of vacua in families ([FBZ, §10.3.1] for a single smooth pointed curve, and [Cod, Prop. 3.6] for families of stable curves, see also [DGT1, Thm 5.1]), and the fact that given simple V-modules M and N, one has

$$\mathrm{rank}\,\mathbb{V}_0(V;M,N,V) = \delta_{M,N'}. \tag{2}$$

This follows from the identification of the three-point, genus zero conformal blocks for simple V-modules M, N, and W with the vector space $\mathrm{Hom}_{A(V)}(A(W) \otimes_{A(V)} M_0, N_0)$ [FZ, Li]. Here $A(V)$ is Zhu's semisimple associative algebra assigned to V, and $A(W)$ is an $A(V)$-bimodule generalizing the algebra $A(V)$ for a V-module W [FZ]. For $W = V$, the space of conformal blocks is thus isomorphic to $\mathrm{Hom}_{A(V)}(M_0, N_0)$. The assumption that M and N are simple V-modules gives that M_0 and N_0 are simple $A(V)$-modules, and thus (2) follows from Schur's lemma. We note that statements asserting (2) can be found in case $V \cong V'$ or for even lattice vertex algebras in [Hua, FHL, DL], and (2) is implicitly assumed elsewhere in the literature.

Finally, the CohFT is semisimple, or equivalently, the Frobenius algebra $\mathcal{F}(V)_{\mathbb{C}}$ is semisimple. This follows from the general argument in [Bea, §5 and Prop. 6.1] using contragredient duality in genus zero, non-negativity of ranks, and the factorization property. Contragredient duality, that is,

$$\operatorname{rank} \mathbb{V}_0\left(V; M^1, \ldots, M^n\right) = \operatorname{rank} \mathbb{V}_0\left(V; (M^1)', \ldots, (M^n)'\right), \tag{3}$$

is obtained in the affine Lie algebra case as a consequence of the stronger statement [Bea, Prop. 2.8]. In our case, we proceed as follows: By propagation of vacua, we can assume that enough of the modules M^i are equal to V, and by the factorization property, we can reduce to compute the rank over a totally degenerate stable rational curve, such that each component has at least one marked point where the adjoint module V is assigned. Then (3) follows from (2). □

3.2. **Computing the TQFT.** As a consequence of Theorem 1, the ranks of vector bundles of coinvariants of vertex algebras of CohFT-type form a semisimple TQFT. Hence all ranks are determined by the dimensions of vector spaces of coinvariants on a rational curve with three marked points. We describe here the reconstruction of the TQFT of the ranks from the fusion rules following results on semisimple TQFTs [Tel, LV].

Let $\mathcal{F}(V)_{\mathbb{C}}$ be the fusion ring given from the semisimple TQFT determined by a vertex algebra V of CohFT-type, as in §2.1. Let $\{e_i\}_i$ be a *semisimple basis* of $\mathcal{F}(V)$, that is, $\eta(e_i, e_j) = \delta_{i,j}$ and $e_i * e_j = \delta_{i,j}\lambda_i e_i$, for some $\lambda_i \in \mathbb{C}$. The values λ_i are known as the *semisimple values* of the TQFT. Let $\{e^i\}_i$ be the dual basis to $\{e_i\}_i$.

Proposition 3.2.1. *The ranks of the vector bundles of coinvariants on $\overline{\mathcal{M}}_{g,n}$ assigned to n finitely-generated modules over a vertex algebra V of CohFT-type is given by the following linear functional on $\mathcal{F}(V)^{\otimes n}$:*

$$\sum_i \lambda_i^{2(g-1)+n} \underbrace{e^i \otimes \cdots \otimes e^i}_{n\,\text{times}}.$$

The statement is a special case of [LV, Prop. 4.1] applied to our TQFT. Examples are given in §5. In the case of coinvariants from modules over affine Lie algebras, the above formula reproduces the classical *Verlinde numbers* after some algebraic manipulations (see e.g., [Bea, Gol]).

4. Chern classes on $\overline{\mathcal{M}}_{g,n}$

In this section we prove Corollary 1 following [MOP+2]. We work with a vertex algebra V of CohFT-type. In particular the set $\mathscr{W}$ of simple V-modules is finite. The Chern characters of bundles of coinvariants are given by the polynomial $P_V(a_\bullet)$ defined as a sum over stable graphs. We start by reviewing stable graphs below, and then define the contributions to $P_V(a_\bullet)$ corresponding to vertices, edges, and legs.

4.1. **Stable graphs and module assignments.** A *stable graph* is the dual graph of a stable curve. We only recall the basic features; for more details see, e.g., [PPZ]. A stable graph Γ comes with a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$, a half-edge set $H(\Gamma)$, and a leg set $L(\Gamma)$. Each leg has a label $i \in \{1,\dots,n\}$, and this gives an isomorphism $L(\Gamma) \cong \{1,\dots,n\}$. Each edge $e \in E(\Gamma)$ is the union of two half-edges $e = \{h,h'\}$, with $h,h' \in H(\Gamma)$. Each vertex $v \in V(\Gamma)$ has a genus label $g_v \in \mathbb{Z}_{\geq 0}$ and a valence n_v counting the number of half-edges and legs incident to v. The genus of the graph Γ is defined as $\sum_{v\in V(\Gamma)} g_v + h^1(\Gamma)$, where $h^1(\Gamma)$ is the first Betti number of Γ.

A stable graph Γ of genus g with n legs identifies a locally closed stratum in $\overline{\mathcal{M}}_{g,n}$ equal to the image of the glueing map of degree $|\mathrm{Aut}(\Gamma)|$:

$$\xi_\Gamma\colon \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g_v,n_v} =: \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}.$$

Given a stable graph Γ, a *module assignment* is a function of type

$$\mu\colon H(\Gamma) \longrightarrow \mathscr{W}$$

such that for $(h,h') \in E(\Gamma)$, one has $\mu(h') = \mu(h)'$, that is, $\mu(h')$ is the contragredient module of $\mu(h)$ (§1.4).

4.2. **Vertex contributions.** Fix a stable graph Γ and a module assignment $\mu\colon H(\Gamma) \to \mathscr{W}$. To each vertex v of Γ is assigned a collection of simple V-modules $M^{i_1},\dots,M^{i_{n_v}}$, one for each leg or half-edge incident to v: for each leg i incident to v, the module M^i is assigned to v; and for each half-edge h incident to v, the module $\mu(h)$ is assigned to v. The vertex contribution is defined as

$$\mathrm{Cont}_\mu(v) := \mathrm{rank}\, \mathbb{V}_{g_v}\left(V; M^{i_1},\dots,M^{i_{n_v}}\right).$$

4.3. **Edge contributions.** Fix a stable graph Γ and a module assignment $\mu\colon H(\Gamma) \to \mathscr{W}$. For each edge $e = \{h,h'\}$, let $a_{\mu(h)}$ be the conformal dimension of $\mu(h)$. The edge contribution is defined as

$$\mathrm{Cont}_\mu(e) := \frac{1 - \mathrm{e}^{a_{\mu(h)}(\psi_h+\psi_{h'})}}{\psi_h + \psi_{h'}}.$$

This is well defined since $\mu(h)$ and $\mu(h')$ have equal conformal dimension.

4.4. **The polynomial $P_V(a_\bullet)$.** Following the computations of [MOP$^+$2], consider the following polynomial with coefficients in $H^*\big(\overline{\mathcal{M}}_{g,n}\big)$:

$$P_V(a_\bullet) := \mathrm{e}^{\frac{c}{2}\lambda} \sum_{\Gamma,\,\mu} \frac{1}{|\mathrm{Aut}(\Gamma)|} (\xi_\Gamma)_* \left(\prod_{i=1}^{n} \mathrm{e}^{a_i \psi_i} \prod_{v\in V(\Gamma)} \mathrm{Cont}_\mu(v) \prod_{e\in E(\Gamma)} \mathrm{Cont}_\mu(e) \right),$$

where c is equal to the central charge of V. The sum in the formula is over all isomorphism classes of stable graphs Γ of genus g with n legs, and over all module assignments μ. For degree reasons, the exponentials are finite sums, and $P_V(a_\bullet)$ is indeed a polynomial.

For the vector bundle of coinvariants of modules over an affine vertex algebra, [MOP$^+$2] shows that $\mathrm{ch}\,(\mathbb{V}(V;M^\bullet)) = P_V(a_\bullet)$, where $V = L_\ell(\mathfrak{g})$ is the simple affine vertex algebra, and M^i is a simple $L_\ell(\mathfrak{g})$-module of conformal dimension a_i, for each i. We extend this result to prove Corollary 1:

Proof of Corollary 1. From [MOP$^+$2, Lemma 2.2], a semisimple CohFT is uniquely determined by the restriction of the classes to $\mathcal{M}_{g,n}$. More precisely, the results in [MOP$^+$2] imply that any semisimple CohFT whose restriction to $\mathcal{M}_{g,n}$ is as in Corollary 2.3.1 for some $c, a_i \in \mathbb{Q}$ is given by an expression as in the statement. $\square$

5. Examples and projects

5.1. **Vertex algebras with no nontrivial modules.** We start with coinvariants constructed from a *holomorphic* vertex algebra V of CohFT-type, that is, a vertex algebra of CohFT-type such that V is the unique simple V-module. Any bundle of coinvariants of modules over V on $\overline{\mathcal{M}}_{g,n}$ has rank 1 and first Chern class equal to $\frac{c}{2}\lambda$, where c is the central charge of V. The rank assertion is in Example 2.5.2. The first Chern class follows from Theorem 1, as the conformal dimension of the adjoint module is zero.

Example 5.1.1. There are 71 self-contragredient, holomorphic vertex algebras of CohFT-type with conformal dimension 24. This very special class includes the moonshine module vertex algebra $V^\natural$ (whose automorphism group is the monster group), and the vertex algebra given by the Leech lattice [LS1]. For such $V = \oplus_{i=0}^{\infty} V_i$, the weight one Lie algebra V_1 is either semi-simple, abelian of rank 24, or 0. If V_1 is abelian of rank 24, then V is isomorphic to the Leech lattice vertex algebra. If $V_1 = 0$, it is conjectured that $V \cong V^\natural$. Vertex algebras with the 69 other possible Lie algebras V_1 have been constructed in [LS1]. Each gives a vector bundle of coinvariants of rank 1 and first Chern class 12λ.

5.2. **Chern classes of bundles from even lattice vertex algebras.** As we illustrate below, even lattice vertex algebras are of CohFT-type, hence following Theorem 2.1.1, their simple modules define vector bundles on $\overline{\mathcal{M}}_{g,n}$.

For the definitions of lattice vertex algebras we recommend [Bor, FLM2, Don, LL]. We briefly review the notation. Let L be a positive-definite even lattice. That is, L is a free abelian group of finite rank d together with a positive-definite bilinear form $(\cdot,\cdot)$ such that $(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$. One assigns to L the *even lattice vertex algebra* V_L. This has finitely many simple modules $\{V_{L+\lambda}\,|\,\lambda \in L'/L\}$, where $L' := \{\lambda \in L\otimes_{\mathbb{Z}}\mathbb{Q}\,|\,(\lambda,\mu)\in\mathbb{Z}$, for all $\mu \in L\}$ is the dual lattice [Don]. Contragredient modules are determined by $V'_{L+\lambda} = V_{L-\lambda}$. The following statement follows from results in the literature.

Proposition 5.2.1 ([Bor, FLM1, Don, DL, DLM2])**.** *For a positive-definite even lattice L of rank d, one has:*

i) The lattice vertex algebra V_L is of CohFT-type with central charge $c = d$;
ii) The conformal dimension of the module $V_{L+\lambda}$ is $\min_{\alpha\in L}\frac{(\lambda+\alpha,\lambda+\alpha)}{2}$;
iii) For the fusion algebra $\mathcal{F}(V_L)_{\mathbb{C}} = \oplus_{\lambda\in L'/L}\mathbb{C}h_\lambda$, the product is given by

$$h_{\lambda_1} * h_{\lambda_2} = h_{\lambda_1+\lambda_2}.$$

Proof. By [Bor, FLM1] V_L satisfies property (I), by [Don] V_L is rational, and hence satisfies property (II), and by [DLM2, Proposition 12.5] V_L is C_2-cofinite, so satisfies property (III). The central charge is computed in [FLM1, Theorem 8.10.2], and the conformal dimension is deduced implicitly in [Don, page 260]. The fusion rules are described in [DL, Chapter 12]. □

Proposition 5.2.1 contains all ingredients needed to compute ranks of bundles of coinvariants from modules over V_L applying Proposition 3.2.1 and their Chern characters applying Corollary 1. Let us discuss some examples.

Remark 5.2.2. There are a number of lattices L one may use to construct the vertex algebras V_L, and it is straightforward to cook up a lattice of almost any rank, whose discriminant group L'/L has arbitrary order. The order of the discriminant is the determinant of the Gram matrix for a basis of the lattice. For instance, to obtain $L'/L \cong \mathbb{Z}/2k\mathbb{Z}$, for $k \in \mathbb{N}$, one can take a one-dimensional lattice with basis vector e such that $(e,e) = 2k$.

For any root system (see e.g., [KMRT, pgs 352–355]), there is a root lattice Λ, and for those of type A, D, E, F, and G, the lattice is even, and gives rise to a vertex algebra V_Λ. Every irreducible root system Λ corresponds to a simple Lie algebra $\mathfrak{g}_\Lambda$. If one normalizes the associated bilinear form (encoded by the Dynkin diagram), then in these cases one has $V_\Lambda \cong L_1(\mathfrak{g}_\Lambda)$, the simple affine vertex algebra at level 1 (see [FLM1] and [LL, Rmk 6.5.8] for details). The weight lattice gives the dual lattice Λ'. For instance, for $\Lambda = A_{m-1}$ the root lattice has rank $m-1$, so that $V_\Lambda \cong L_1(\mathfrak{sl}_m)$ has conformal dimension $m-1$.

One may also construct vertex algebras V_L by taking L to be the direct sum of lattices described above, getting quotient lattices L'/L that are of the form $\mathbb{Z}/m_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/m_k\mathbb{Z}$ for arbitrary $m_1,\ldots,m_k$. Such lattices may be interpreted as Mordel-Weil lattices (see e.g., [Shi, SS]).

The root lattices are very special, and one may easily construct more general lattices with discriminant groups isomorphic to $\mathbb{Z}/m\mathbb{Z}$. For instance if m is prime and congruent to 0 or 3 mod 4, this can be done with a rank 2 even lattice (the order of the discriminant group for an even lattice of rank 2 is always congruent to 0 or 3 mod 4). As the diversity of quadratic forms cannot be overstated (see e.g., [Con2, Con1, BH]), there are many potentially interesting classes from lattice vertex algebras.

Example 5.2.3. Let V_L be a vertex algebra given by an even unimodular lattice of rank d. Because the lattice is unimodular, V_L is self-contragredient and it has no nontrivial modules. In particular, any bundle of coinvariants from modules over V_L has rank one and first Chern class $\frac{d}{2}\lambda$.

Example 5.2.4. Consider an even lattice L of rank d with $L'/L \cong \mathbb{Z}/2\mathbb{Z}$. The vertex algebra V_L has two simple modules $V = V_L$ and W. From Proposition 5.2.1, the product in $\mathcal{F}(V_L)_{\mathbb{C}} = \mathbb{C}h_V \oplus \mathbb{C}h_W$ is given by

$$h_V * h_V = h_V, \qquad h_V * h_W = h_W, \qquad h_W * h_W = h_V.$$

With terminology as in §3.2, a semisimple basis for $\mathcal{F}(V_L)_{\mathbb{C}}$ is

$$e_1 := \frac{1}{\sqrt{2}}(h_V + h_W), \qquad e_2 := \frac{1}{\sqrt{2}}(h_V - h_W),$$

with semisimple values both equal to $\sqrt{2}$. One has $h_V = \dfrac{1}{\sqrt{2}}(e_1 + e_2)$ and $h_W = \dfrac{1}{\sqrt{2}}(e_1 - e_2)$. Applying Proposition 3.2.1, the rank of the bundle $\mathbb{V}_g\left(V; V^{\otimes p} \otimes W^{\otimes q}\right)$ on $\overline{\mathcal{M}}_{g,n}$ for $p + q = n$ is

$$\begin{aligned}\operatorname{rank} \mathbb{V}_g\left(V; V^{\otimes p} \otimes W^{\otimes q}\right) &= \sqrt{2}^{\,2g-2+n} \left(\frac{1}{\sqrt{2}^{\,n}} + (-1)^q \frac{1}{\sqrt{2}^{\,n}} \right) \\ &= 2^g \, \delta_{q,\text{ even}}.\end{aligned}$$

In particular, the rank vanishes when q is odd. Applying Corollary 1, when $q = 2r$, the Chern character is

$$\begin{aligned}&\operatorname{ch} \mathbb{V}_g\left(V; V^{\otimes p} \otimes W^{\otimes 2r}\right) = \\ &\qquad \mathrm{e}^{\frac{d}{2}\lambda} \sum_{\Gamma} \frac{2^{g-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} (\xi_\Gamma)_* \left(\prod_{i=1}^{2r} \mathrm{e}^{a\psi_i} \prod_{e \in E(\Gamma)} \frac{1 - \mathrm{e}^{a(\psi_h + \psi_{h'})}}{\psi_h + \psi_{h'}} \right).\end{aligned}$$

Here a is the conformal dimension of W. The sum in the formula is over those isomorphism classes of stable graphs Γ of genus g with n legs such that for each vertex, the number of assigned W at the incident legs is even. Note that the only module assignment μ contributing nontrivially to the polynomial P_V in this case is $\mu\colon h \mapsto W$, for all half-edges h.

Example 5.2.5. Let L be an even lattice such that $L'/L \cong \mathbb{Z}/m\mathbb{Z}$, for $m \geq 2$. Let $\mathscr{W} = \{V = W_0, \dots, W_{m-1}\}$ be the set of simple V_L-modules. The fusion rules from [DL] give

$$\operatorname{rank} \mathbb{V}_0\left(V_L; W_i \otimes W_j \otimes W_k\right) = \delta_{i+j+k \equiv_m 0}.$$

This implies that

$$\operatorname{rank} \mathbb{V}_0 \left(V_L; W_0^{\otimes n_0} \otimes \cdots \otimes W_{m-1}^{\otimes n_{m-1}}\right) = \delta_{\sum_{j=0}^{m-1} j n_j \equiv_m 0}$$

and by induction on the genus and the factorization property, we can further deduce that

$$\operatorname{rank} \mathbb{V}_g \left(V_L; W_0^{\otimes n_0} \otimes \cdots \otimes W_{m-1}^{\otimes n_{m-1}}\right) = m^g \, \delta_{\sum_{j=0}^{m-1} j n_j \equiv_m 0}. \tag{4}$$

Example 5.2.6. One can also obtain the rank found in Example 5.2.5 using Proposition 3.2.1. Namely, for an even lattice L such that $L'/L \cong \mathbb{Z}/m\mathbb{Z}$, a semisimple basis for the fusion ring $\mathcal{F}(V_L)_{\mathbb{C}} = \oplus_{i=0}^{m-1} \mathbb{C} h_i$ is

$$e_i := \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \rho^{ij} h_j, \qquad \text{for } i = 0, \ldots, m-1,$$

where $\rho \in \mathbb{C}$ is a primitive m-th root of unity. One checks $e_i * e_i = \sqrt{m}\, e_i$, hence the semisimple values are all $\sqrt{m}$. As in Example 5.2.4, one applies Proposition 3.2.1 to recover (4).

Remark 5.2.7. Examples 5.2.4–5.2.6 show that ranks of bundles of coinvariants from modules over an even lattice L such that $L'/L \cong \mathbb{Z}/m\mathbb{Z}$ coincide with ranks of bundles of coinvariants from modules over the affine Lie algebra $\widehat{\mathfrak{sl}}_m$ at level one. However, while the ranks depend only on L'/L, the Chern characters depend additionally on the quadratic form of L, responsible for the conformal dimension of the irreducible V_L-modules. It is reasonable to expect that classes from lattice vertex algebras could give a collection of CohFTs larger than the one obtained from the affine Lie algebra case.

5.3. **Commutants and the parafermion vertex algebras.** For a vertex algebra V and a vertex subalgebra U of V, one may construct the *commutant*, or *coset*, vertex algebra $\mathrm{Com}_V(U)$ of U in V [FZ]. It would be interesting to study the Chern classes for bundles of coinvariants of modules over $\mathrm{Com}_V(U)$ for pairs $U \subset V$ such that $\mathrm{Com}_V(U)$ is of CohFT-type.

Conjecturally, if U and V are both of CohFT-type, then $\mathrm{Com}_V(U)$ is also of CohFT-type. However, U and V need not be of CohFT-type: one such example is given by the well-studied family of cosets of the Heisenberg vertex algebra in the affine vertex algebra $L_k(\mathfrak{g})$ for a finite-dimensional simple Lie algebra $\mathfrak{g}$ at level $k \in \mathbb{Z}$. The Heisenberg vertex algebra is not rational, nor C_2-cofinite (see e.g., [FBZ] for a discussion of the Heisenberg vertex algebra). Nevertheless, the *parafermion vertex algebras* are known to be of CohFT-type [DR] (see also [ALY, DLWY, DLY, DW1, DW2, DW3]). These are related to W-algebras [ALY]. The necessary invariants for expressing the Chern classes of bundles of coinvariants of modules over parafermion vertex algebras are known from [DKR, DR, ADJR], and one could proceed as in §5.2.

5.4. **Orbifold vertex algebras.** Let G be a subgroup of the group of automorphisms of a vertex algebra V. The *orbifold vertex algebra* V^G consists of the fixed points of G in V. In case V is of CohFT-type, its full group of automorphisms G is a finite-dimensional algebraic group [DG2]. If G is also solvable, then V^G will also be of CohFT-type [Miy, CM]. Conjecturally, V^G is always of CohFT-type. We note that even if V is holomorphic, and therefore has no non-trivial modules, the vertex algebra V^G will not generally be holomorphic [GK, DVVV, DPR].

One could for instance consider orbifold vertex algebras created from parafermion vertex algebras. In some cases, the simple modules and the fusion rules are known in the literature [JW1, JW2, JW2].

Similarly, one could construct orbifold vertex algebras starting from lattice vertex algebras. In [BE], simple modules for orbifolds V_L^G, where G is generated by an isometry of order two, are classified and their fusion rules are given. Explicit examples with root lattices and Dynkin diagram automorphisms are given in [BE, §4].

6. Questions

Summarizing, bundles of coinvariants defined by modules over vertex algebras of CohFT-type share three important properties with their classical counterparts: They (i) support a projectively flat logarithmic connection [TUY, DGT1]; (ii) satisfy the *factorization property*, a reflection of their underlying combinatorial structure [TUY, DGT2]; and (iii) give rise to cohomological field theories, as we show here.

As described in Remark 5.2.2, bundles of coinvariants from lattice vertex algebras are generalizations of those given by affine Lie algebras at level one. It is natural to expect that other known properties of the classical case extend to the vertex algebra case, and a number of questions come to mind.

Question 1. Given a simple, simply connected algebraic group G with associated Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ and $\ell \in \mathbb{Z}_{>0}$, the simple vertex algebra $V = L_\ell(\mathfrak{g})$ is of CohFT-type [FZ, DLM2]. By [BL, Fal, KNR], for a smooth algebraic curve C, there is a natural line bundle D on the moduli stack $\mathrm{Bun}_G(C)$ of G-bundles on C such that, for any point P in C, there is a canonical isomorphism

$$\mathbb{V}\left(L_\ell(\mathfrak{g}); L_\ell(\mathfrak{g})\right)^\dagger_{(C,P)} \cong H^0\left(\mathrm{Bun}_G(C), D^\ell\right),$$

where $L_\ell(\mathfrak{g})$ is the adjoint module over itself. By [Pau, LS2], given V-modules $M^\bullet$, there is a line bundle L on the moduli stack of quasi-parabolic G-bundles $\mathrm{ParBun}_G(C, P_\bullet)$, for which $\mathbb{V}(V; M^\bullet)^\dagger_{(C,P_\bullet)}$ is isomorphic to the global sections of L on $\mathrm{ParBun}_G(C, P_\bullet)$. This geometric picture holds for stable curves with singularities as well [BF] (see also [BG1]).

The automorphism group $\mathrm{Aut}(V)$ of a vertex algebra V of CohFT-type is a finite-dimensional algebraic group [DG2]. The connected component $\mathrm{Aut}(V)^0$ of $\mathrm{Aut}(V)$ containing the identity has been described in a number of cases [DGR, DG1, DG3, DG2]. For instance for $V = L_\ell(\mathfrak{g})$, one has $\mathrm{Lie}\left(\mathrm{Aut}(V)^0\right) \cong V_1 \cong \mathfrak{g}$ [DG2]. For V of CohFT-type, can one find a geometric realization for conformal blocks defined from modules over V, for instance involving algebraic structures on curves related to $\mathrm{Aut}(V)^0$? Ideas in this direction have been considered in [Uen] and [BZF].

Question 2. Gromov-Witten invariants of smooth projective homogeneous spaces define base-point-free classes on $\overline{\mathcal{M}}_{0,n}$; divisors defined from Gromov-Witten invariants of $\mathbb{P}^r = \mathbb{G}r(1, r+1)$ are equivalent to first Chern classes of bundles from integrable modules at level one over $\mathfrak{sl}_{r+1}$ [BG2, Props 1.4 and 3.1]. Numerical evidence suggests a more general connection between classes of bundles at level ℓ with Gromov-Witten divisors for Grassmannians $\mathbb{G}r(\ell, r+\ell)$ [BG2]. By Witten's Dictionary, the quantum cohomology of Grassmannians can be used to compute ranks of conformal blocks bundles in type A for any level [Bel]. Are there connections between other Gromov-Witten theories and the more general bundles of coinvariants studied here?

Question 3. Vector bundles defined by representations of affine Lie algebras are globally generated in genus zero, and so Chern classes have valuable positivity properties. For instance, first Chern classes are base-point-free, giving rise to morphisms [Fak]. Can one give sufficient conditions on vertex algebras of CohFT-type and their modules so that the vector bundles of coinvariants are globally generated? Chern classes of bundles from certain Virasoro vertex algebras are not nef, so further assumptions must be made.

Question 4. Bundles of coinvariants from affine Lie algebras give rise to morphisms from $\overline{\mathcal{M}}_{0,n}$ to Grassmannian varieties [Fak]. In the special case where the Lie algebra is of type A and modules are at level one, we know the image varieties parametrize configurations of weighted points on rational normal curves in projective spaces [GS, Gia, GG]. If Chern classes given by representations of particular types of vertex algebras are base-point-free, can one give modular interpretations for the images of their associated maps?

Question 5. Classes associated to bundles of coinvariants on $\overline{\mathcal{M}}_{0,n}$ defined by $V = L_\ell(\mathfrak{g})$ satisfy scaling and level-rank identities, and are zero above a critical level, allowing one to give sufficient conditions for when they lie on extremal faces of the nef cone [AGS, BGM2, BGM1]. Do Chern classes studied here satisfy similar identities? Can one find criteria to ensure they lie on extremal faces of cones of nef cycles? Do bundles defined by particular modules over vertex operator algebras generate extremal rays? Bundles from vertex operator algebras constructed from exotic lattices may be relevant.

Question 6. In [BG3, Theorem 5.1], using factorization, Verlinde bundles constructed from level one integrable modules over $\mathfrak{sl}_{r+1}$ are shown to be isomorphic to both GIT bundles [BG3], and to the r-th tensor power of cyclic bundles studied in [Fed]. Are there other such identifications, for instance involving the line bundles of coinvariants on $\overline{\mathcal{M}}_{0,n}$ constructed from even lattice theories, as discussed in Example 5.2.5?

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