On factorization & vector bundles of conformal blocks from vertex algebras

Angela Gibney

Rutgers University, New Brunswick

2020

This work was done with with

Chiara Damiolini and Nicola Tarasca.

We consider vector bundles of coinvariants: Each fiber is a vector space of coinvariants, derived from a pointed curve and a collection of "nice" modules over a certain type of conformal vertex algebra.

The news is that the vector spaces of coinvariants satisfy the factorization property. This fact is one of the crucial ingredients used to show that the vector spaces fit together to form vector bundles. In fact, these bundles owe many of their nice features to the factorization property. I will list some of these properties here today.

Vector bundles of coinvariants on $\overline{\mathcal{M}}_{g,n}$

$$\begin{array}{ccc} \mathbb{V}_g(V;\mathbf{M}^\bullet) & \qquad & \mathbb{V}_g(V;\mathbf{M}^\bullet)_{(C,P_\bullet)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & & (C,P_\bullet) \end{array}$$

fibers are vector spaces of coinvariants

$$\mathbb{V}_g(V;\mathbf{M}^\bullet)_{(C,P_\bullet)} := [\mathbf{M}^\bullet]_{L_{(C,P_\bullet)}}.$$

Vector spaces of coinvariants are quotients

Vector spaces of coinvariants are quotients

For

- $(C, P_\bullet)$ a stable n-pointed curve; and
- $M^1, \ldots, M^n$ finitely generated admissible modules over a conformal vertex algebra V.

the vector space of coinvariants

$$[\mathbf{M}^\bullet]_{L_{(C,P_\bullet)}} = \mathbf{M}^\bullet / L_{(C,P_\bullet)}(V) \cdot \mathbf{M}^\bullet$$

is the largest quotient of the tensor product

$$\mathbf{M}^\bullet = \otimes_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{M}^{\mathrm{i}}$$

by the action of a Lie algebra

$$L_{(C,P_\bullet)}(V).$$

To define these quotients, even informally as I will, one must describe the Lie algebra $L_{(C,P_\bullet)}(V)$, and how it acts on the tensor product of the modules $\mathbf{M}^\bullet = \otimes_{i=1}^n M^i$.

There are two Lie algebras that act.

Surprisingly, their coinvariants are isomorphic (DGT2).

Before defining these quotients, and the notions that go into them, I briefly state our main results.

Main Theorem (DGT I, DGT 2, DGT 3)

For $M^1, \ldots, M^n$ finitely generated admissible modules over a conformal vertex algebra such that V is

(a) of CFT type, (b) rational, and (c) C_2-cofinite,

then vector spaces of coinvariants

1. *are finite dimensional;*
2. *satisfy factorization; and*
3. *are fibers of a vector bundle $\mathbb{V}_g(V; \mathbf{M}^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$.*
4. *$\mathbb{V}_g(V; \mathbf{M}^\bullet)$ carries a projectively flat connection.*
5. *The Chern character of $\mathbb{V}_g(V; \mathbf{M}^\bullet)$ gives rise to a semi-simple cohomological field theory.*

We call a conformal vertex algebra V satisfying (a) of CFT type, (b) rational, and (c) C_2-cofinite, a vertex algebra of CohFT-type.

I'll come back to this later in the talk.

Our results extend prior work for:

- affine Lie algebras and generalizations;
- the Virasoro, and generalizations;

As I'll illustrate, we obtain a number of new examples.

Results for sheaves from integrable modules over affine Lie algebras

(1987) Tsuchiya and Kanie: $(\mathbb{P}^1, P_\bullet)$ with coordinates.

(1991) Tsuchiya, Ueno, and Yamada: coordinatized $(C, P_\bullet) \in \overline{\mathcal{M}}_{g,n}$; sheaves satisfy (1), (2), (3), & (4).

(1993) Tsuchimoto: the vector bundles and the connection descend to $\overline{\mathcal{M}}_{g,n}$.

(2010) Fakhruddin: glob. gen. and first Chern class.

(2015) Marian, Oprea, Pandharipande: first Chern class in the tautological ring (different construction).

(2017) Building on MOP, Marian, Oprea, Pandharipande, Pixton, and Zvonkine show (5).

I'm leaving out a lot of work on these bundles (eg results about the connection and their classes, including the Verlinde formula).

Following MOP, to save space on slides, I will refer to them as Verlinde bundles.

Zhu's coinvariants (generalizing (TUY))

(1994) Zhu defined coinvariants and conjectured factorization for quasi primary generated (qpg) V.

(2005) Abe and Nagatomo proved (1) for C smooth, and qpg V satisfying (a) & (c).

(2005) Nagatomo and Tsuchiya (2005) show (2) & (3) for $g = 0$, $V \cong V'$, different hypothesis, extending Zhu's coinvariants for $g = 0$ curves with singularities.

(2005) Huang showed (2) for $g \in \{0, 1\}$, $V \cong V'$ satisfying (a),(b) & (c).

(2019) Codogni showed (2) for all g, for holomorphic $V \cong V'$ satisfying (a),(b) & (c).

Notes: (1) If $V \cong V'$, then V is qpg. (2) Zhu's coinvariants generalize those studied by TUY.

Notes:

(1) I haven't said enough to define qpg, but I will say that a VOA of CFT type is qpg if and only if $V \cong V'$. I will mention this later when I talk about the Lie algebras themselves (FHL 1993), (DLM 1996, 1997). (2)

Just as for the Verlinde bundles, I have left out a ton of work about these, especially with regard to conformal field theory and the important results of my colleague Yi Zhi Huang.

Virasoro & FBZ coinvariants

(1991) Beilinson, Feigin, and Mazur construct coinvariants and prove factorization for coordinatized curves and the Virasoro VOA.

(2004) Frenkel and BenZvi define sheaves of coinvariants for coordinatized points $(C, P_\bullet) \in \mathcal{M}_{g,n}$ and modules over conformal vertex algebras, generalizing (BFM). They show sheaves support a projectively flat connection and mention that factorization is expected if V is rational.

(2019) We extend coinvariants (and the connection) to stable curves, showing independent of coordinates (DGT1). We prove factorization, and using factorization with the connection, show the sheaves are locally free (DGT2).

New Examples

Vertex algebras of CohFT-type include:

- Positive definite even lattice VOAs V_L;
- The moonshine module $V^\natural$;
- Parafermions; and
- certain W-algebras;

New VOAs of CohFT-type from old ones:

1. tensor products: $V^1, \ldots, V^k$ of CohFT type then $V^1 \otimes \cdots V^k$ is too;
2. commutants/cosets; and
3. orbifolds.

The two latter types are more complicated.

Commutant and coset examples

Commutant or Coset
For U a vertex subalgebra of V, conjecturally, if U and V are both of CohFT-type, then $\mathrm{Com}_V(U)$ is also of CohFT-type.

Orbifold Let $G \subset Aut(V)$. The *orbifold vertex algebra* V^G consists of the fixed points of G in V. If V is of CohFT-type, $G = Aut(V)$ is a finite-dimensional algebraic group. If G is also solvable, then V^G will also be of CohFT-type. Conjecturally, V^G is always of CohFT-type.

All the examples we know are self-contragredient.

All the examples we know are self-contragredient.

Question

If V is of CohFT-type, is $V \cong V'$?

All the examples we know are self-contragredient.

Question

If V is of CohFT-type, is $V \cong V'$?

The Chiral/FBZ Lie algebra is defined for any conformal V, whereas Zhu's Lie algebra only for V qpg. We have proved factorization in case V is of CohFT type, which is presumably in a more general setting than Zhu conjectured.

For V of CFT-type then $V \cong V'$ if and only if V is qpg (references (FHL, and DLM 1996, 1997)). A yes to this question implies if V CohFT-type then V is qpg. In particular, factorization would hold exactly where Zhu indicated it should.

OK, they have good properties, but why study these vector bundles?

OK, they have good properties, but why study these vector bundles?

Here are three reasons, based on what is known to be true for the Verlinde bundles:

1. They may provide new examples of rational conformal field theories.
2. They give rise to elements in the important tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$, and therefore may be useful in testing Pixton's conjectures.
3. They may be interesting in terms of the birational geometry of the moduli space of curves.

A little bit about the tautological ring

The tautological ring is a subring of the Chow ring generated by "tautological classes"

Defined with generators in the '80s, Mumford suggested finding all relations.

- Faber and Zagier did so for $\mathcal{M}_g$ (FZ).
- Pixton (P) conjectured relations generalizing (FZ).
- Work of Pandharipande, Pixton, Zvonkine (2014) and of Janda (2015) give that classes obtained from the Verlinde bundles and CohFTs of a similar nature obey Pixton's relations (PPZ 2014).

Chern classes of the bundles $\mathbb{V}_g(V; \mathbf{M}^\bullet)$ lie in the tautological ring.

Question

Do they obey Pixton's relations?

Global generation

If $\mathbb{V}_g(V; M^\bullet)$ is globally generated
$\implies c_1(\mathbb{V}_g(V; M^\bullet))$ is base point free.

If $c_1(\mathbb{V}_g(V; M^\bullet))$ is base point free, it follows from (GKM, 2003) that one must have that certain coefficients in the first Chern class are all nonnegative. These coefficients are invariants of the vertex algebra and of the modules used to define the bundles. This nonnegativity condition doesn't hold for all the bundles we consider but it does hold for Verlinde bundles (Fakhruddin, 2011).

Question

Can one specify conditions so that the bundles $\mathbb{V}_g(V; M^\bullet)$ *are generated by their global sections?*

One guess is that V should be unitary.

For the remainder of the talk I will briefly describe the terms used to define coinvariants, state the factorization theorem, and the idea of the proof.

Brief incomplete definition (mainly for notation)

A conformal vertex algebra is a tuple $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$, where

- $V = \oplus_{i \in \mathbb{N}} V_i$ is a $\mathbb{C}$-vector space, with $dim V_i < \infty$;
- $\mathbf{1}^V \in V_0$ (the vacuum vector),
- $\omega \in V_2$ (the conformal vector);
- $Y(\cdot, z)\colon V \to End(V)[[z, z^{-1}]]$ is a linear function assigning to every element $A \in V$ the *vertex operator*

$$Y(A, z) := \sum_{i \in \mathbb{Z}} A_{(i)} z^{-i-1}.$$

The datum $(V, \mathbf{1}^V, \omega, Y(\cdot, z))$, referred to as V, must satisfy a number of axioms.

To give an example, the conformal structure comes from coefficients of the vertex operators

$$Y(\omega, z) = \sum_{i \in \mathbb{Z}} \omega_{(i)} z^{-i-1}.$$

Endomorphisms $L_p := \omega_{(p+1)}$, are subject to the Virasoro relations, giving the action of a Virasoro Lie algebra on V

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p-q)\,\omega_{(p+q+1)} + \frac{c}{12}\,\delta_{p+q,0}\,(p^3 - p)\,\mathbf{id}_V.$$

Here $c \in \mathbb{C}$ is the *central charge* of V. Moreover:

$$\omega_{(1)}|_{V_i} = i \cdot \mathbf{id}_V, \text{ for all } i, \text{ and } Y\left(\omega_{(0)}A, z\right) = \partial_z Y(A, z).$$

Admissible V-modules

Admissible V-modules

For V a conformal vertex algebra, an admissible V-module is a pair $(M, Y^M(\,-\,, z))$ where M is an $\mathbb{N}$-graded vector space,

$$Y^M(\cdot, z)\colon V \to \mathbf{End}(M)[[z, z^{-1}]],$$

$$A \mapsto Y^M(A, z) := \sum_{i\in\mathbb{Z}} A^M_{(i)} z^{-i-1},$$

and for $A \in V_d$,

$$A^M{}_{(i)} M_j \subset M_{j+d-i-1}.$$

This pair must satisfy a number of conditions analogous to the VOA axioms.

If V is a conformal vertex algebra such that

(a) V is of CFT type

$V \cong \bigoplus_{i \in \mathbb{N}} V_i$, and $V_0 \cong \mathbb{C}$;

(b) V is rational

finitely generated admissible modules
are completely reducible;

(c) V is C_2-cofinite

$$\dim(V/C_2(V)) < \infty,$$
$$C_2(V) = \mathrm{Span}_{\mathbb{C}}\{A_{(-2)}B : A, B \in V\}.$$

then we say V is of CohFT-type,

Vertex algebras of CohFT-type have some very good properties, that make our jobs easier.

- If V is rational (or V is C_2-cofinite) there are just finitely many simple modules.
- If V is rational and C_2-cofinite, the simple admissible modules are the same as the simple ordinary modules (DLM, 1997 Remark 2.4).
- Ordinary modules are extraordinary, satisfying additional finiteness conditions, including
  1. graded pieces M_λ are finite dimensional
  2. for fixed λ, one has $M_{\lambda+\ell} = 0$ for $\ell >> 0$.

We next define the Lie algebras and their actions on modules. To describe the actions we first introduce the “ancillary Lie algebra".

Ancillary Lie algebra

Ancillary Lie algebra

Given a pointed curve (C, P), and t a local parameter on C at P, let

$$\mathcal{L}_P(V) = V \otimes \mathbb{C}((t))/Im\nabla,$$

where $\nabla : V \otimes \mathbb{C}((t)) \longrightarrow V \otimes \mathbb{C}((t))$, is the map

$$A \otimes f \mapsto L_{-1}A \otimes f + A \otimes \frac{d}{dt}f.$$

Generators and relations

$\mathcal{L}_P(V)$ has generators

$$\overline{A \otimes t^j} := A_{[j]} \in \mathcal{L}_P(V) = V \otimes \mathbb{C}((t))/Im\nabla,$$

and relations

$$[A_{[j]}, B_{[k]}] = \sum_{\ell \geq 0} \binom{j}{\ell} (A_{(\ell)}(B))_{[j+k-\ell]}.$$

$\mathcal{L}_P(V)$ acts on $\otimes_i M^i$

For M^i a V-module "at $P_i \in C$"

$$\bigoplus_{i=1}^{n} \mathcal{L}_{P_i}(V) \times \otimes_i M^i \to \otimes_i M^i,$$

$$((\ldots, A_{[k_j]}, \ldots), (m_1 \otimes \ldots \otimes m_n)) \mapsto$$

$$\sum_{j=1}^{n} \cdots \otimes m_{j-1} \otimes A_{k_j}^{M^j}(m_j) \otimes m_{j+1} \otimes \cdots .$$

The vertex algebra bundle

To give an algebro-geometric view of $\mathcal{L}_P(V)$, we will use the vertex algebra bundle

$$\mathcal{V}_C \to C,$$

defined for a smooth curve C by Frenkel and BenZvi, and extended in (DGT1) to stable curves with singularities. The fiber over a a point $P \in C$ is (non-canonically) isomorphic to V (which is an infinite object, so this is non-standard).

One can show that the bundle supports a connection

$$\nabla : \mathcal{V}_C \to \mathcal{V}_C \otimes \omega_C.$$

and

Theorem (FBZ, DGT1)

For $U = C \setminus \cup P_i$ affine

$$(\mathcal{V}_C \otimes \omega_C)|_U \cong \bigoplus_j V_j \otimes H^0(U, \omega_C^{1-k}).$$

(This result is only important if later after my talk, you ask about the definition of Zhu's algebra).

Algebro-geometric view of $\mathcal{L}_P(V)$

For the punctured disc

$$D_p^X = Spec(Frac(\mathcal{O}_{C,p}/m_{C,p})),$$

one has that:

Theorem (FBZ, DGT 1)

$$\mathcal{L}_P(V) \cong H^0(D_p^x, \mathcal{V}_C \otimes \omega_C / Im\nabla).$$

FBZ/Chiral Lie algebra

Given a stable pointed curve $(C, P_\bullet)$, set

$$\mathcal{L}_{(C,P_\bullet)}(V) := H^0(C \setminus \cup P_i, \mathcal{V}_C \otimes \omega_C / Im\nabla).$$

One can show that the restriction

$$H^0(C \setminus \cup P_i, \mathcal{V}_C \otimes \omega_C / Im\nabla) \longrightarrow \bigoplus_j H^0(D^\times_{P_j}, \mathcal{V}_C \otimes \omega_C / Im\nabla),$$

$$\sigma \mapsto (\sigma|_{D^\times_{P_1}}, \ldots, \sigma|_{D^\times_{P_n}})$$

is a map of Lie algebras.

Diagonal action by restriction.

$$\mathcal{L}_{(C,P_\bullet)}(V) \times \otimes_i M^i \to \otimes_i M^i,$$

defined by

$$(\sigma, m_1 \otimes \cdots \otimes m_n) \mapsto \sum_{j=1}^{n} \cdots \otimes m_{j-1} \otimes \sigma|_{D^X_{P_j}} \cdot m_j \otimes \cdots .$$

The vector space of coinvariants

$$\mathbb{V}_{\mathfrak{g}}(V; \mathbf{M}^\bullet)_{(C,P_\bullet)} := [M^\bullet]_{\mathcal{L}_{(C,P_\bullet)}(V)}.$$

is the largest quotient of the tensor product $\otimes_i M^i$ on which $\mathcal{L}_{(C,P_\bullet)}(V)$ acts trivially.

Factorization

Factorization

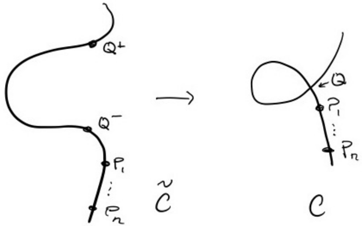
Factorization enables one to transform fibers of the bundle to fibers defined on simpler curves. This leads to recursions, and allows one to make inductive arguments. This is the crucial ingredient allowing ranks of such bundles, and their Chern characters to be given explicitly.

Theorem (Factorization)

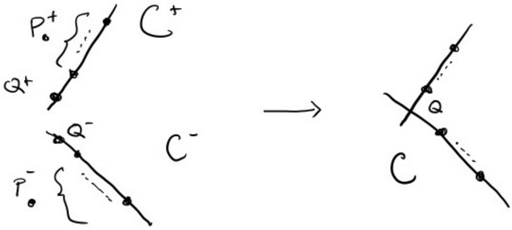
For V a conformal vertex algebra of CohFT-type, then with the notation as in the picture

$$\mathbb{V}(V; M^{\bullet})_{(C,P_{\bullet})} \cong \bigoplus_{W \in \mathcal{W}} \mathbb{V}(V; M^{\bullet} \otimes W \otimes W')_{(\widetilde{C}, P_{\bullet} \sqcup Q_{\bullet})}.$$

For $\mathcal{W}$ all simple V-modules, the normalization $\widetilde{C} \to C$, $Q^+, Q^- \in \widetilde{C}$ preimages of a node Q, and $Q_{\bullet} = (Q^+, Q^-)$.



For a curve with a separating node:



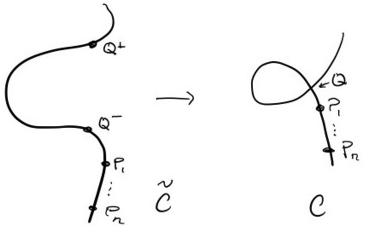
$$\mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{(\widetilde{C}, P_\bullet \sqcup Q_\bullet)} \cong \mathbb{V}\left(V; M_+^\bullet \otimes W\right)_{X^+} \otimes \mathbb{V}\left(V; M_-^\bullet \otimes W'\right)_{X_-}$$

where $X^\pm = (C^\pm, P_\bullet^\pm \sqcup Q^\pm)$, and $M_\pm^\bullet$ are the modules at the $P_\bullet^\pm$ on $C^\pm$.

The idea of the proof of factorization

The idea of the proof of factorization

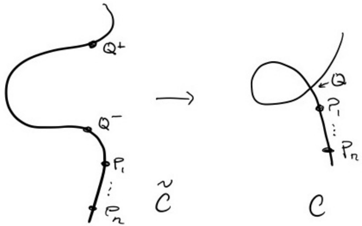
Consider the normalization of a pointed curve C with a non-separating node Q



The idea going back to TUY is to insert a trivial module Z at the two points of $\tilde{C}$ lying over Q so coinvariants remain the same (trivial modules don't effect coinvariants).

The idea of the proof of factorization

Consider the normalization of a pointed curve C with a non-separating node Q



The idea going back to TUY is to insert a trivial module Z at the two points of $\widetilde{C}$ lying over Q so coinvariants remain the same (trivial modules don't effect coinvariants). This almost works.

We obtain a diagram

$$\begin{array}{ccc} [M^\bullet \otimes Z]_{\mathcal{L}_{(\widetilde{C},P_\bullet \cup Q_\bullet)}(V)} & \xleftarrow{h} & [M^\bullet]_{\mathcal{L}_{(\widetilde{C},P_\bullet)}(V,D)} \\ \downarrow & & \downarrow \\ [M^\bullet \otimes \overline{Z}]_{\mathcal{L}_{(\widetilde{C},P_\bullet \cup Q_\bullet)}} & \xrightarrow{\cong} & [M^\bullet]_{\mathcal{L}_{(C,P_\bullet)}} \end{array} .$$

Finite dimensionality of the fibers is important to our argument. Taking duals, we work with vector spaces of conformal blocks via correlation functions.

The End (thank you).

A question somebody might ask

Question

How does one define Zhu´s Lie algebra?

As mentioned earlier, in order to define vector spaces of coinvariants, one could have used Zhu's Lie algebra, as long as V is quasi-primary generated.

A quasi-primary vector is an element $A \in V$ such that $L_1 A = 0$. One says that V is quasi-primary generated (qpg) if V is generated by quasi-primary vectors.

V is qpg if and only if $L_1 V_1 = 0$ (DLM 1996).
If V is qpg then it is easier to define maps involving V as local charts patch together more easily. This comes into the definition of Zhu's Lie algebra, which is defined as the image of a map which is a Lie algebra map in case V is qpg.

In particular, we set

$$\mathfrak{g}_{C\setminus P_\bullet}(V) := \varphi_{\mathfrak{g}}\left(\oplus_{k\geq 0} V_k \otimes H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right)\right)$$

where

$$\varphi_{\mathfrak{g}}\colon\ \oplus_{k\geq 0} V_k \otimes H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) \to \oplus_{i=1}^{n} \mathcal{L}_{P_i}(V)$$

is the map induced by

$$B \otimes \mu \mapsto \left(\mathrm{Res}_{t_i=0}\, Y[B, t_i]\mu_{P_i}(dt_i)^k\right)_{i=1,\ldots,n}.$$

In the previous slide, t_i is a formal coordinate at the point P_i, $Y[B, t_i] := \sum_{k \in \mathbb{Z}} B_{[k]} t_i^{-k-1}$, and μ_{P_i} is the Laurent series expansion of μ at P_i, the image of μ via

$$H^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) \to H^0\left(D_{P_i}^\times, \omega_C^{\otimes 1-k}\right) \simeq_{t_i} \mathbb{C}((t_i))(dt_i)^{1-k}.$$

When V is qpg with $V_0 \cong \mathbb{C}$, Zhu shows that $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is a Lie subalgebra of $\mathfrak{L}(V)^{\oplus n}$.