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Moduli spaces reveal how objects like varieties or schemes of a particular type behave in families. For instance, the moduli space $\overline{\mathcal{M}}_{g,n}$, parametrizing stable n-pointed curves of genus $g$, gives insight into the study of smooth curves and their degenerations. As curves arise in many contexts, the moduli space of curves is a meeting ground, where constructions from diverse fields can be tested and explored.

A natural goal when studying a space, like $\overline{\mathcal{M}}_{g,n}$, is to characterize the maps admitted by it. From this perspective, vector bundles are important, as their sections can be thought of as functions.

My recent work concerns vector bundles on $\overline{\mathcal{M}}_{g,n}$ defined from representations of vertex operator algebras, generalizing Verlinde bundles, defined by integrable modules over affine Lie algebras. In earlier research, I studied aspects the moduli space of curves, using Mori theory and tropical geometry.

Here I briefly describe a selection of my results, arranged according to the following themes:

(1) Bundles on the moduli space of curves from modules over vertex algebras;
(2) Verlinde bundles and the geometry of the moduli space of curves;
(3) Moduli, reductions, and generalizations of the F-Conjecture.

## 1. Bundles on the moduli space of curves from modules over conformal vertex algebras

For years it has been understood that coinvariants defined from the action of a Lie algebra on modules $M^i$ over a conformal vertex algebra satisfying certain properties give rise to sheaves $\mathbb{V}_g(V;M^\bullet)$ on moduli of *smooth* n-pointed curves of genus $g$ [FBZ04]. In the special cases of modules over Virasoro vertex algebras, and of integrable modules over affine Lie algebras, the sheaves were shown to extend to stable curves with singularities, and have particularly nice properties [TK88, TUY89, BDF91]. For instance, they were shown to be locally free of finite rank, giving vector bundles. Those vector bundles were shown to satisfy factorization (and the more refined sewing property), a feature that makes recursive arguments about ranks and Chern classes possible. For the latter type, defined from integrable modules over affine Lie algebras, the sheaves are sometimes referred to as Verlinde bundles as their ranks can be computed with the Verlinde formula [MOP15, MOP$^+$17]. The Verlinde bundles support a projectively flat connection, and their Chern characters determine cohomological field theories [TUY89, Tsu93, MOP15, MOP$^+$17] .

There have been questions as to whether the more general sheaves of modules over conformal vertex algebras could be extended to stable curves with singularities, and if so, what conditions would ensure they satisfy the valuable properties known to hold for the Verlinde bundles [Zhu94, FBZ04]. Results for $g = 0$, and $g = 1$ gave support to speculation and conjecture regarding what should suffice [AN03a, AN03b, NT05, Hua05a, Hua05b, Hua05c, Hua08].

With my coauthors C. Damiolini and N. Tarasca, I have identified three properties that if satisfied by a conformal vertex algebra $V$ guarantee that all features mentioned above hold. Namely, if $V$ is (I) of CFT-type; (II) rational; and (III) $C_2$-cofinite, then sheaves of coinvariants defined from finitely generated admissible modules extend to the boundary, are locally free of finite rank, satisfy factorization, support a projectively flat connection, and give way to cohomological field theories:

*Sheaves on $\overline{\mathcal{M}}_{g,n}$ and their projectively flat connection.* In [DGT19a], we show that coinvariants of finitely generated admissible modules $M^i$ over a conformal vertex algebra satisfying properties (I), (II), and (III) give rise to a quasi-coherent sheaf $\mathbb{V}_g(V;M^\bullet)$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of *stable* n-pointed curves of genus $g$. We also show the sheaves $\mathbb{V}_g(V;M^\bullet)$ carry a twisted logarithmic D-module structure, and hence support a projectively flat connection.

*Vector bundles and factorization.* In [DGT19b] we show that vector spaces of coinvariants defined by finitely generated admissible modules $M^i$ over a conformal vertex algebra satisfying properties (I), (II), and (III) satisfy factorization, and the sewing theorem. As an application, we prove the sheaves $\mathbb{V}_g(V;M^\bullet)$ are locally free and of finite rank, giving vector bundles on $\overline{\mathcal{M}}_{g,n}$.

*Vertex algebras of CohFt-type.* In [DGT19c] we show that, as for Verlinde bundles, those given by finitely generated admissible modules over a conformal vertex algebra satisfying properties (I), (II), and (III) give rise to cohomological field theories. Following [MOP$^+$17], we give a polynomial expression (with rational coefficients) for their Chern characters. In this paper we say that if they satisfy properties (I), (II), and (III), then the vertex algebra is of CohFT-type.

*Future directions.* Given the similarities shown between bundles of coinvariants defined by finitely generated admissible modules over a conformal vertex algebra of CohFT-type with Verlinde bundles, it is very likely that they share other important properties. We hope to prove that for $g = 0$, and $g > 0$, $n >> 0$, if defined by unitary modules, bundles of coinvariants defined by conformal vertex algebras of CohFT-type are globally generated. This could lead to information about the birational geometry of the moduli space of curves, as it has for Verlinde bundles. We also hope to find a connection between generalized theta functions, and vector spaces of conformal blocks.

## 2. Verlinde bundles and the geometry of the moduli space of curves

Coinvariants defined by integrable modules over affine Lie algebras define vector bundles on the moduli space of curves, sometimes called Verlinde bundles as their ranks can be computed with Verlinde formula. These have special properties: Dual spaces to fibers have connections to generalized theta functions, and in genus zero bundles are known to be globally generated, and so their Chern classes have positivity properties. Some of my results about these objects are briefly described below.

*Basepoint free Gromov-Witten loci on $\overline{\mathcal{M}}_{0,n}$.* In [BG18], with P. Belkale, we study basepoint free loci on $\overline{\mathcal{M}}_{0,n}$ defined from the Gromov-Witten invariants of smooth projective homogeneous spaces $X$. When $X$ is projective space, divisors are shown equivalent to first Chern classes of Verlinde bundles defined by integrable modules at level one over simple Lie algebras in type A.

*Finite generation of the determinate of cohomology line bundle.* For $\mathrm{G}$ a simple, simply connected complex linear algebraic group, $C$ stable of genus $g \geq 2$, $\mathrm{Bun}_{\mathrm{G}}(C)$ is the stack of principal G-bundles on $C$. To any representation $\mathrm{G} \to \mathrm{Gl}(V)$, let $\mathcal{D}(V)$ on $\mathrm{Bun}_{\mathrm{G}}(C)$, be the determinant of cohomology line bundle. In [BG], we show, when G is $\mathrm{SL}(r)$, and for the standard representation $\mathrm{SL}(r) \to \mathrm{Gl}(V)$, one has that $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D}(V)^{\otimes m})$ is finitely generated. The was known for smooth curves [BL94, Fal94]. As an application, using Verlinde bundles, we show there is a flat family over $\overline{\mathcal{M}}_g$ whose fibers are normal and projective, and whose fibers over smooth $C$ are moduli spaces $\mathrm{SU}_C(r)$, parameterizing semistable vector bundles of rank $r$ with trivial determinant on $C$.

*Geometric interpretations at points on the boundary.* At smooth curves, conformal blocks for affine Lie algebras correspond to generalized theta functions [BL94, Fal94, KNR94, Pau96, LS97], and in [BG, Theorems 1.3 and 1.4], with P. Belkale, we show this is true for singular curves as well, in prescribed circumstances. In [BGK16], with P. Belkale and A. Kazanova, we show using intersection theory on $\overline{\mathcal{M}}_g$, this is not true in all situations.

*Vanishing and identities.* Characteristic classes of Verlinde bundles are subject to a number of types of identities, which can give information about their associated maps. For instance, in [BGM15], with P. Belkale and S. Mukhopadhyay, we found new rank and level identities satisfied by first Chern classes of bundles defined by integrable modules over affine Lie algebras of type $A$ *at the critical level*, generalizing work of Fakhruddin in type $A_1$. Inspired by unexpected examples instances of trivial divisors, in [BGM16] we introduce the problem of finding necessary and sufficient conditions to determine when the first Chern class is nontrivial (solving the problem in type $A_1$). We also show under certain hypothesis, first Chern classes are subject to additive identities. Identities for first Chern classes were also found in [AGSS12], and [AGS14], and in [GM16], extended to higher Chern classes.

*Birational models.* Verlinde bundles have helped to provide modular interpretations of images of birational contractions of $\overline{\mathcal{M}}_{0,n}$, including GIT quotients generalizing Kapranov's compactifications of $\mathrm{M}_{0,n}$ [Kap93a, Kap93b], generalized by Giansiracusa and Simpson. In [GG12] and [GJMS13] with my coauthors Giansiracusa, and Giansiracusa, Jensen, Moon, and Swinarski, this is further developed.

*Finite generation of full-dimensional subcones of the Nef cone.* In [GG12], with Giansiracusa, we show the infinite set of first Chern classes on $\overline{\mathcal{M}}_{0,n}$ of bundles defined by integrable modules at level one over affine Lie algebras in type $A$ spans a finitely generated, full-dimensional subcone of the cone of nef divisors. This was generalized in different contexts to full dimensional cones generated by infinite sets of first Chern classes of rank one bundles by [Kaz16], my former postdoc, and [Hob19], my student.

*Conformal blocks and the Torelli map from $\overline{\mathcal{M}}_g$ to certain compactifications of $A_g$.* While not known to be base point free generally, conformal blocks divisors can sometimes be used to study moduli of higher genus curves. In [Gib12], I identify one of the faces of the nef cone of $\overline{\mathcal{M}}_g$, consisting of semi-ample divisors, showing they are pullbacks of ample divisors along the Torelli map from $\overline{\mathcal{M}}_g$ to certain types of compactifications of $A_g$, the moduli space of Abelian varieties of dimension g.

## 3. Generalizations of the moduli of curves, Tropical compactifications and the F-conjecture.

*Pointed trees of projective spaces.* In [CGK09], Chen, Krashen and I construct a space $\mathrm{T}_{d,n}$, whose points correspond to $n$-pointed stable rooted trees of $d$-dimensional projective spaces, which for $d = 1$, are $(n + 1)$-pointed stable rational curves. Smooth and projective, with boundary a smooth normal crossings divisor, the space $\mathrm{T}_{d,n}$ has an inductive construction analogous to but differing from Keel's construction of $\overline{\mathcal{M}}_{0,n}$. We describe its Chow groups, Chow ring, Chow motive, and Poincaré polynomials, and use this information to answer a question of Fulton and MacPherson.

*Reduction of the F-Conjecture to genus zero.* The F-Conjecture asserts that a divisor on $\overline{\mathcal{M}}_{g,n}$ is nef if and only if it nonnegatively intersects one of a finite number of rational curves on $\overline{\mathcal{M}}_{g,n}$ called F-Curves. In particular, the F-Conjecture implies that $\mathrm{Nef}(\overline{\mathcal{M}}_{g,n})$ is polyhedral, the convex hull of a finite number of extremal rays.

In [GKM02], we proved that the F-Conjecture on $\overline{\mathcal{M}}_{0,g+n}$ implies the F-Conjecture on $\overline{\mathcal{M}}_{g,n}$. The conjecture holds for low $g$ and $n$ [KM13, GKM02, FG03, Gib09].

*Reduction of the F-Conjecture to log canonical.* In [Gib09], I reduce the $F-$conjecture on $\overline{\mathcal{M}}_{g,n}$ to showing that certain divisors in $\overline{\mathcal{M}}_{0,N}$ for $N \leq g+n$ are equivalent to the sum of the canonical divisor plus an effective divisor supported on the boundary. As an application, I give numerical criteria, which if satisfied by a divisor $D$ on $\overline{\mathcal{M}}_g$, show that $D$ is nef. Using computer software, written by Krashen, this is used to verify the conjecture for $g \leq 24$.

*Generalization to tropical compactifications.* While generally not a toric variety, it is possible to embed $\overline{\mathcal{M}}_{0,n}$ into a noncomplete toric variety $X_\Delta$, and in [GM10], we give equations for $\overline{\mathcal{M}}_{0,n}$ in the Cox ring of $X_\Delta$. In [GM12], we study varieties $X$ that, like $\overline{\mathcal{M}}_{0,n}$, can be embedded in a toric variety $X_\Delta$, as a tropical compactification. We use $X_\Delta$ to define cones which are upper and lower bounds for $\mathrm{Nef}(X)$. For $X = \overline{\mathcal{M}}_{0,n}$, we show that our upper bound cone $U(\overline{\mathcal{M}}_{0,n})$, is the cone of divisors predicted by the F-Conjecture to be equal to $\mathrm{Nef}(\overline{\mathcal{M}}_{0,n})$. In other words, this result generalizes the $F$-Conjecture on $\overline{\mathcal{M}}_{0,n}$ to varieties that like $\overline{\mathcal{M}}_{0,n}$, are tropical compactifications.

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