BASEPOINT FREE CYCLES ON $\overline{\mathrm{M}}_{0,n}$ FROM GROMOV-WITTEN THEORY

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Abstract. Basepoint free cycles on the moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed rational curves, defined using Gromov-Witten invariants of smooth projective homogeneous spaces are introduced and studied. Intersection formulas to find classes are given. Gromov-Witten divisors for projective space are shown equivalent to conformal blocks divisors for type A at level one.

In this work we use the Gromov-Witten theory of smooth projective homogeneous spaces to produce a seemingly rich source of effective cycles of arbitrary codimension on the moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed rational curves. These classes are nef and basepoint free, and so for instance in codimension one their associated rational maps are in fact morphisms.

One application of this construction is a surprising identification with Chern classes of vector bundles of conformal blocks, also known in the literature as bundles of covacua: In Theorem 3.1 we show that Gromov-Witten divisor classes for $\mathbb{P}^r = Gr(1, r+1)$ coincide with conformal blocks divisors for $\mathfrak{sl}_{r+1}$ at level one. We have noticed in examples, and in terms of parameters, there may be a more general connection between Gromov-Witten divisors for Grassmannians $Gr(\ell, r+\ell)$ and conformal blocks divisors for $\mathfrak{sl}_{r+1}$ at higher levels ℓ. In Question 3.3 we propose a specific relationship between divisor classes.

Theorem 3.1 can be seen as a generalization of the type of pairing in Witten's theorem, relating quantum cohomology of Grassmannians to ranks of conformal blocks bundles in type A, and geometric methods of producing invariants from Schubert calculus (see the survey [Bel10], and recent work [RZ18]). Theorem 3.1 relates two families of classes defined from seemingly different perspectives: one enumerative, and the other in terms of the representation theory of affine Lie algebras. This echoes correspondences between classes arising in other contexts (eg. [PP17, MNOP06, GV98]).

The Gromov-Witten type loci we consider here have interesting properties from the perspective of natural cones of positive cycles, and in particular are always basepoint free. This can be seen intuitively from their construction. Informally, following [KM94, FP97], they consist of points in $\overline{\mathrm{M}}_{0,n}$ which are the images of pre-stable curves admitting a stable map of some particular degree to a smooth projective homogeneous variety and such that the images of the marked points lie on some fixed Schubert subvarieties in general position[1]. Since the target variety is homogeneous, by moving the Schubert varieties via the group action, one can show using Kleiman's transversality theorem that the associated linear system does not have a base locus.

Not only do they reside in the nef cone, but one can give general criteria indicating when classes lie on extremal faces (eg see Propositions 4.4, 4.5, where we have done this for odd quadrics). As an application, we give an example in Section 4.3 of a Gromov-Witten divisor defined from an odd quadric that spans an extremal ray of the nef cone of $\overline{\mathrm{M}}_{0,n}$ for all even n. We have demonstrated in examples for projective space, and even quadrics in low rank that there are are Gromov-Witten classes that lie on extremal faces of the nef cone not known to be spanned by conformal blocks divisors (see Sections 3.2 and 4.5). This illustrates that the Gromov-Witten classes are more amenable to

[1] All definitions, and requirements are explained in Section 0.1.

computational investigation than conformal blocks divisors, where the rank of the bundle can slow a computer down.

Gromov-Witten divisors have other advantages over conformal blocks divisors: With no known "modular" description for them, it is difficult to explicitly describe conformal blocks divisors in a linear system. In contrast, the Gromov-Witten divisors parameterize loci, and properties of the associated morphisms should therefore be more accessible. So while the computational edge allows one to see Gromov-Witten divisors that are not known to be given by Chern classes of conformal blocks bundles, there are benefits to finding identities between Gromov-Witten divisors and conformal blocks divisors, as the former add more to the picture.

We show in Proposition 1.4 that the Gromov-Witten type loci defined here satisfy a more robust and functorial basepoint free condition, closed under intersection products up to rational equivalence, which we call *rational strongly basepoint free* after Fulger and Lehmann [FL17] (see Definition 1.1). We often call these strongly basepoint free. In Lemma 1.3, we verify that in this context, the pushforward of strongly basepoint free classes along flat maps are strongly basepoint free. Since forgetful maps $\overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,m}$ with $m < n$ are flat, one obtains basepoint free classes of codimension k on $\overline{\mathrm{M}}_{0,n}$ by pushing forward strongly basepoint free classes of higher codimension (like the GW classes). So for example, higher codimension classes are useful even if one is only interested in divisors.

Underlying these results is the explicit determination of cycle classes, which is done by intersecting with a dual basis. For instance, a divisor class on $\overline{\mathrm{M}}_{0,n}$ is computed by intersecting with boundary curves. Expressions for such an intersection are given in Propositions 2.2 and 2.3. In practice, to compute the classes, one needs (1) The small quantum cohomology rings of the homogeneous spaces X, and (2) Four point big quantum cohomology numbers (where the underlying pointed curve is not held fixed in the enumerative problem).

In case the rational cohomology of X is generated by divisors, as for $X = G/B$ with B a Borel subgroup, and G a semisimple algebraic group, the formula in Proposition 2.2 can be simplified considerably. In particular, as is shown in Proposition 2.5, it follows from [KM94] that four point big quantum cohomology numbers can be reduced recursively to the computation of small quantum cohomology numbers.

It would of course be very interesting to characterize images of maps given by Gromov-Witten divisors. In Section 5 two questions for classes defined from Gromov-Witten invariants for non-homogeneous spaces are considered.

0.1. Gromov-Witten theory preliminaries.

We work entirely over the field of complex numbers. The set of all stable maps of genus g and degree $\beta \in H_2(X)$, with n marked points to a normal variety X forms a (Deligne-Mumford) moduli stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Stable maps are tuples $((C,\vec{p}),f)$, where $(C,\vec{p})$ is a pre-stable curve and f is a stable map from $(C,\vec{p})$ to X. A pre-stable curve $(C,\vec{p})$ is a connected, complete, and reduced curve C of genus g, with at worst nodal singularities, and $\vec{p} = (p_1,\dots,p_n)$ a finite collection of n smooth points on C. A stable map is any morphism from a pre-stable curve to X such that there are only finitely many automorphisms of the map.

The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{virt} \in A_{\nu,\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$ is constructed in [BF97,LT98]. The virtual dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is

$$\nu = (3g-3+n) + c_1(T_X)\cdot\beta + (1-g)\dim X. \tag{0.1}$$

Definition 0.1. *Consider homogeneous cycle classes $\alpha_1,\dots,\alpha_n \in A^*(X)$, with $\alpha_i \in A^{|\alpha_i|}(X)$, $\beta \in H^2(X)$ and $c \geq 0 \in \mathbb{Z}$. We say that a triple $(X,\beta,\vec{\alpha})$ satisfies the* **codimension c cycle**

condition *in genus g if*

$$c = \sum_{i=1}^{n} |\alpha_i| - c_1(T_X)\cdot\beta - (1-g)\dim X. \tag{0.2}$$

Definition 0.2. *Consider homogeneous cycle classes $\alpha_1,\dots,\alpha_n \in A^*(X)$, with $\alpha_i \in A^{|\alpha_i|}(X)$, $\beta \in H^2(X)$, and $c \geq 0 \in \mathbb{Z}$. Assume first that a triple $(X,\beta,\vec{\alpha})$ satisfies the codimension c cycle condition in genus g. Let*

$$I^{c,X}_{g,\beta,\vec{\alpha}} = \eta_*\big(\prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{virt}\big) \in A^c_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}) \tag{0.3}$$

where $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ are the n evaluation maps, $\eta : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ is the contraction map, and $\alpha_i \in A^{|\alpha_i|}(X)$ for $1 \leq i \leq n$.

If $(X,\beta,\vec{\alpha})$ does not satisfy the codimension c cycle condition in genus g, define $I^{c,X}_{g,\beta,\vec{\alpha}} = 0 \in A^c_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$.

We note that $I^{0,X}_{g,\beta,\vec{\alpha}} \in \mathbb{Q} = A^0_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$. To ease notation, we will drop g in the notation, and when the context is clear we write $I^{c,X}_{g,\beta,\vec{\alpha}} = I^{c,X}_{\beta,\vec{\alpha}}$. We note that for

$$e = \nu - \sum_{i=1}^{n}|\alpha_i| = (3g-3+n) + c_1(T_X)\cdot\beta + (1-g)\dim X - \sum_{i=1}^{n}|\alpha_i|,$$

one has

$$I^{c,X}_{g,\beta,\vec{\alpha}} \in A_{e,\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}) = A^c_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}),$$

where of course $c = \dim \overline{\mathcal{M}}_{g,n} - e$.

Remark 0.3. *In this work we assume that $g = 0$, and so we for the most part make the substitution $I^{c,X}_{0,\beta,\vec{\alpha}} = I^{c,X}_{\beta,\vec{\alpha}}$. Except for in Section 5, we will assume that X is a homogeneous space. In* [KP01] *it is shown that for homogeneous spaces X, the (coarse) moduli space for $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is connected, for any genus g. Since in genus zero, for homogeneous X, the moduli space $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is locally the quotient of a nonsingular variety by a finite group, connectedness is equivalent to irreducibility, in this case we may work with the plain fundamental class.*

While we don't use them, localization techniques are often employed to compute these invariants in many cases, especially for homogeneous X [GP99].

0.2. **Methods for obtaining basepoint free cycles on $\overline{\mathrm{M}}_{0,n}$.** An effective cycle α of codimension k is basepoint free if the base locus of α is empty:

Definition 0.4. *Let Z be an arbitrary variety. A cycle $\alpha \in A_k(Z)$ is basepoint free if for any point $z \in Z$, there is an effective k-cycle β on Z, rationally equivalent to α, such that z is not in the support of β.*

The following methods are known for obtaining basepoint free cycle classes on $\overline{\mathrm{M}}_{0,n}$:

1. Chern classes (and Schur polynomials in the Chern classes) of conformal block bundles, also called bundles of coinvariants (covacua) (see §1.3).
2. Gromov-Witten classes $I^{c,X}_{\beta,\vec{\alpha}}$ with X homogeneous (see Proposition 1.4).
3. Algebraic operations in (1) and (2): intersection products, pushforwards under point-dropping maps $\overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,m}$, with $m < n$, and iterations of these (Proposition 1.4 and Lemma 1.3).

Remark 0.5. *Both Gromov-Witten classes and Chern characters of the conformal block bundles give rise to cohomological field theories* [KM94, Pan17]*, and it would be interesting to know how they compare to one another.*

1. Rational strongly basepoint freeness and GW-cycles

Here we define the notion of rational strongly basepoint free cycles, which is inspired by the one given in [FL17] for strongly basepoint free cycles. In Lemma 1.3 we list a number of properties satisfied by such strongly basepoint free cycles. In Proposition 1.4, we show that Gromov-Witten cycles $I^{c,X}_{\beta,\vec{\alpha}}$ with X homogeneous, are rationally strongly basepoint free. In Remark 1.7, we point out that Schur polynomials in the Chern classes of $\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)$ are rationally strongly basepoint free on $\overline{\mathrm{M}}_{0,n}$.

Recall that forgetful maps $\overline{\mathrm{M}}_{0,n}\to\overline{\mathrm{M}}_{0,m}$ with $m<n$ are flat. Lemma 1.3 together with Propositions 1.4 and 1.7 therefore are a source of basepoint free cycles on the moduli spaces $\overline{\mathrm{M}}_{0,n}$. In particular, one obtains basepoint free classes on $\overline{\mathrm{M}}_{0,n}$ by pushing forward strongly basepoint free classes of higher codimension on suitable $\overline{\mathrm{M}}_{0,n'}$ with $n'>n$.

1.1. Rational strongly basepoint free cycles.

Definition 1.1. *A Chow cycle $\alpha\in A^k(X)$ of codimension k on an equidimensional possibly singular, reducible, and/or disconnected projective variety X is said to be* rationally strongly basepoint free *if there is a flat morphism $s:U\to X$ from an equidimensional quasi-projective scheme U and a proper morphism $p:U\to W$ of relative dimension* $\dim X-k$*, where W is a variety isomorphic to an open subset of $\mathbb{A}^m$ for a suitable m, such that each component of U surjects onto W, and $\alpha=(s|F_p)_*[F_p]$, where F_p is a general fiber of p (hence α can be represented by an effective cycle).*

Definition 1.2. *Denote the semigroup of rationally strongly basepoint free classes of codimension k on a (possibly singular) projective variety X by* $\mathrm{SBPF}^k(X)\subseteq A^k(X)$.

For rationally strongly basepoint free cycles, unlike for the strongly basepoint free cycles of [FL17], one considers cycles up to rational equivalence, rather than up to numerical equivalence. In case these are different, $\mathrm{SBPF}^k(X)\subseteq A^k(X)$, one would not necessarily form the closure of the cones generated by such classes. We have included the condition that W is an open subset of $\mathbb{A}^m$ since we are interested in rational equivalence. Moreover, one can drop the condition that each component of U surjects onto W since we may replace W by a non-empty open subset, and U by the inverse image of this open set.

Note that if F_{p_i}, $i=1,2$ are two fibers then $(s|F_{p_i})_*[F_{p_i}]$ coincide in $A^k(X)$. Indeed, since p is proper and U and W are quasi-projective, U sits inside a projective space $\mathbb{P}\times W$ over W. Let $\overline{W}$ be a projective space containing W as an open subset. Form the closure $\overline{U}$ of U in the projective variety $\mathbb{P}\times\overline{W}\times X$. We have maps $\overline{U}\to X$ (which may not be flat) and $\overline{U}\to\overline{W}$. Over $W\subseteq\overline{W}$, U and $\overline{U}$ coincide. Therefore F_{p_i} are also fibers of $\overline{U}\to\overline{W}$ and are hence rationally equivalent. Now $\overline{U}\to X$ is proper and hence the pushforwards of the fibers agree in Chow groups.

Lemma 1.3. *Rationally strongly basepoint free classes satisfy the following properties:*

(a) *A rationally strongly basepoint free class $\alpha\in\mathrm{SBPF}^k(Z)$ is basepoint free in the following stronger sense: Given any irreducible subvariety $V\subset Z$ (for example a point), there is a effective cycle of class α which intersects V in no more than expected dimension (if the intersection is non-empty).*

(b) *If Z is a smooth projective variety and $\alpha\in\mathrm{SBPF}^k(Z)$ and $\beta\in\mathrm{SBPF}^{k'}(Z)$, then their intersection product $\alpha\cdot\beta\in\mathrm{SBPF}^{k+k'}(Z)$.*

(c) *Let* $\pi : X \to Y$ *be a flat morphism of relative dimension* d *and* $\alpha \in \mathrm{SBPF}^k(X)$*, then* $\pi_*\alpha \in \mathrm{SBPF}^{k-d}(Y)$*.*
(d) *If* X, Y *are projective varieties, with* Y *smooth, and* $\pi : X \to Y$ *is a morphism, then* $\pi^* \mathrm{SBPF}^k(Y) \subseteq \mathrm{SBPF}^k(X)$*.*
(e) *The cycle class of a Schubert variety on a* G/P *is rationally strongly basepoint free. Therefore all effective cycles on a homogeneous space are rationally strongly basepoint free.*
(f) *Let* $\mathbb{V}$ *be a globally generated vector bundle of rank* n *on a smooth projective variety* X*. The Schur polynomial* $s_\lambda = \det(c_{\lambda_i+j-i})_{1\le i,j\le n}$ *in the Chern classes* $c_i = c_i(\mathbb{V})$ *of* $\mathbb{V}$ *lies in* $\mathrm{SBPF}^{|\lambda|}(X)$*. Here* $|\lambda| = \sum |\lambda_i|$ *is the length of the partition* $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$*. See* [FL17, Def 3.2] *and the proof of* [FL17, Lemma 5.7]*.*
(g) *Basepoint free divisors on a smooth variety are rationally strongly basepoint free.*

Proof. For (a), $\dim(V \cap s(F_p)) \le \dim(s^{-1}(V) \cap F_p)$, which in turn is the generic dimension of fibers of $s^{-1}(V) \to W$ which is $\dim U - \dim X + \dim V - \dim W = \dim V - k$. Part (b) follows from [FL17, Corollary 5.6]. The W for the intersection cycle is the product of the W for α and β and is hence rational. Part (c) follows from the same proof as [FL17, Lemma 5.3] (here the W is unchanged for the pushforward): In particular, it isn't necessary to assume that X or Y are smooth. For (d) see [FL17, Lemma 5.4] (particularly the first paragraph of the proof there, the W is unchanged here). In particular, one does not need smoothness of X. Property (e) follows by taking $W = G/B$ (which is rational), U the universal Schubert variety in $G/B \times G/P$, and $X = G/P$. Statement (f) was proved for strongly basepoint free cycles (see [FL17, Def 3.2] and the proof of [FL17, Lemma 5.7]) for smooth varieties, is true for singular projective varieties X as well (using properties (d) and (e) with Y a Grassmannian, noting that Schur polynomials represent cycle classes of Schubert varieties). For (g), note that any basepoint free divisor is the pull back, from a projective space $\mathbb{P}^n$, of an effective divisor by a morphism and hence properties (d) and (e) apply. □

1.2. **GW classes are rationally strongly basepoint free.** Let G be a semisimple complex algebraic group. Let B be the Borel subgroup corresponding to a fixed Cartan decomposition of G. For the rest of the paper we assume $g = 0$, and until §5 we assume X is a homogeneous variety on which the group G acts transitively. It follows that $X = G/P$ where $P \supseteq B$ is a parabolic subgroup.

Let W be the Weyl group of G and W_P the Weyl group of P. For every $w \in W/W_P$ there is a Schubert variety $X_w \subset X$ obtained as the closure of BwP/P. The cycle classes of the Schubert varieties give a $\mathbb{Z}$ basis of $A^*(X)$, we sometimes denote the cycle class corresponding to $w \in W/W_P$ simply as $w \in A^*(X)$.

Now suppose α_i are cycle classes of Schubert varieties $[X_{w_i}]$, $i = 1, \dots, n$. By [FP97], the coarse moduli space $\overline{\mathrm{M}}_{0,n}(X, \beta)$ is equidimensional of the expected dimension (0.1), and we may work with the fundamental class of the coarse moduli space $\overline{\mathrm{M}}_{0,n}(X, \beta)$ instead of the virtual fundamental class. The classes $I^{c,X}_{\beta,\vec{\alpha}}$ are therefore integral Chow cycles.

Proposition 1.4. *Assume* $X = G/P$*, and let* $(X, \beta, \vec{\alpha})$ *satisfy the codimension* c *cycle condition. Then the Gromov-Witten cycle* $I^{c,X}_{\beta,\vec{\alpha}}$ *is a rationally strongly basepoint free on* $\overline{\mathrm{M}}_{0,n}$*, i.e.,* $I^{c,X}_{\beta,\vec{\alpha}} \in \mathrm{SBPF}^k(\overline{\mathrm{M}}_{0,n})$*.*

To prove Proposition 1.4, we refer to the following.

Lemma 1.5. *Let* $\eta : \overline{\mathrm{M}}_{0,n}(X, \beta) \to \overline{\mathrm{M}}_{0,n}$*, and* $x \in \overline{\mathrm{M}}_{0,n}$*.*

(1) *Each component of* $\eta^{-1}(x)$ *has dimension equal to* $\dim \overline{\mathrm{M}}_{0,n}(X, \beta) - \dim \overline{\mathrm{M}}_{0,n}$*;*
(2) *The map* η *is flat.*

Proof. (of Lemma 1.5) *Part (1):* This is of course well known, and follows from [KM94] and [FP97]. For a fixed nodal curve C of arithmetic genus 0, the space of maps $C \to X$ has dimension $\dim X + \beta \cdot T_X = \dim \overline{\mathrm{M}}_{0,n}(X,\beta) - \dim \overline{\mathrm{M}}_{0,n}$ [FP97, Section 5.2]. We have to therefore account for the collapsing operation in which a C has a component which is mapped on to X with positive degree, and has only two special points (the point in $\overline{\mathrm{M}}_{0,n}$ collapses this component). Such maps are subject to a non-trivial equivalence: The extra component has positive dimensional space of automorphisms fixing the marked points, and hence brings down the count of space of maps by at least one.

Part (2): Locally $\overline{\mathrm{M}}_{0,n}(X,\beta)$ is the quotient of a smooth variety Y by a finite group G. The composite map $Y \to Y/G \subseteq \overline{\mathrm{M}}_{0,n}(X,\beta) \xrightarrow{\eta} \overline{\mathrm{M}}_{0,n}$ is flat by [Mat89, Theorem 23.1] since Y and $\overline{\mathrm{M}}_{0,n}$ are smooth and all fibers have the expected dimension by Lemma 1.5. The coordinate ring of Y/G is a direct summand of the coordinate ring of Y, and hence is flat over $\overline{\mathrm{M}}_{0,n}(X,\beta)$ [KV99, Remark 2.6.8]. □

Proof. (of Proposition 1.4) One has maps $\mathrm{ev} : \overline{\mathrm{M}}_{0,n}(X,\beta) \to X^n$ and flat maps $\eta : \overline{\mathrm{M}}_{0,n}(X,\beta) \to \overline{\mathrm{M}}_{0,n}$. We claim that the pull back under ev of $\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n$ is strongly basepoint free. This claim implies Proposition 1.4, since the Gromov-Witten cycle $I^{c,X}_{\beta,\vec{\alpha}}$ is the pushforward $\eta_*(\mathrm{ev}^*(\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n))$ and η is flat (and using Property (c) in Section 1).

Every effective cycle on a projective homogeneous space is strongly basepoint free, see Lemma 1.3 (e). Therefore $\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n$ is a basepoint free cycle on X^n.

We now use the following property (d) of Lemma 1.3: If X, Y are projective varieties, with Y smooth, and $\pi : X \to Y$ is a morphism, then $\pi^* \mathrm{SBPF}^k(Y) \subseteq \mathrm{SBPF}^k(X)$. □

Remark 1.6. *It is easy to see, rather immediately, that* $I^{c,X}_{\beta,\vec{\alpha}}$ *is basepoint free on* $\overline{\mathrm{M}}_{0,n}$. *Let* P *be a point of* $\overline{\mathrm{M}}_{0,n}$, *and let* $Z \subset X^n$ *be the product of Schubert varieties* X_i *with cycle classes* α_i. *Note that* G^n *acts transitively on* X^n. *By Kleiman's Bertini theorem* [Kle74], *for general* $\vec{h} = (h_1, \ldots, h_n) \in G^n$, *one has that* $\mathrm{ev}^{-1}(\vec{h}Z)$ *has the expected codimension inside* $\overline{\mathrm{M}}_{0,n}(X,\beta)$, *and meets the fiber* $\eta^{-1}(P)$ *(which is equidimensional) in the expected dimension, which is easily computed to be* $-c < 0$. *The cap product* (0.3) *is represented by the effective cycle* $\mathrm{ev}^{-1}(\vec{h}Z)$, *and the basepoint freeness follows.*

1.3. Chern classes of conformal blocks on $\overline{\mathrm{M}}_{0,n}$ are rationally strongly basepoint free.

The name conformal blocks bundles, also called bundles of coinvariants (covacua), refers to vector bundles of coinvariants $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ defined on $\overline{\mathcal{M}}_{g,n}$, where $(\mathfrak{g}, \vec{\lambda}, \ell)$ is a compatible triple consisting of a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ are dominant weights for $\mathfrak{g}$ at level ℓ. These were originally constructed in [TUY89]. See [Fak12] for a summary, as well as a proof of global generation in case $g = 0$ (which implies basepoint freeness of the Chern classes), and many relevant examples and results, including formulas for the Chern classes in genus zero (see Remark 1.7 and Lemma 1.3; also see [Ful98, Example 12.1.17]). The total Chern character was given in [MOP+17] in arbitrary genus. In [GM16] some results on higher Chern classes of conformal blocks bundles are discussed.

Remark 1.7. *Schur polynomials in the Chern classes of* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *on* $\overline{\mathrm{M}}_{0,n}$ *are rationally strongly basepoint free. Note that these Schur classes include Chern classes of* $\mathbb{V}$. *Indeed, the vector bundles* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *defined on* $\overline{\mathrm{M}}_{0,n}$, *are globally generated in case* $g = 0$ [Fak12], *and parts (e),(f) of Lemma 1.3 therefore apply. Chern classes of* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *are elements of Fulger and Lehman's Pliant cone, which sits inside the cone of strongly basepoint free divisors. Here it makes sense to talk about the closed cone of rationally strongly basepoint free divisors as numerical and rational equivalence are the same.*

2. GW cycles

In Proposition 2.2 we give a formula for the intersection of a GW cycle of codimension one with F-Curves, described below in Def 2.1. These curves can be used to compute the class of a divisor (see Section 2.1.2). Ingredients for the proof of Proposition 2.2 will be defined in Section 2.2.1. The proof is given in Section 2.2.3.

Prop 2.2 is generalized in Proposition 2.3 to give an explicit formula for the intersection of a GW-loci $I^{c,X}_{\beta,\vec{\alpha}}$ of arbitrary codimension c with a boundary cycle of complementary codimension, which like F-curves, are products of moduli spaces. The proof of Prop 2.2 that of Proposition 2.2, and we state them separately for clarity, and because we focus on divisors.

We show in Section 2.3 how it is sometimes possible to simplify the formulas by reducing four-point classes to three points.

2.1. **Computing classes of GW cycles by intersecting with boundary classes.**

2.1.1. *Intersecting GW divisors with boundary curves.*

Definition 2.1. *If $N_1 \cup \cdots \cup N_4$ is a partition of $[n] = \{1, \dots, n\}$ consisting of four nonempty subsets, then given four points $(\mathbb{P}^1, \{p_i\}_{i\in N_j} \cup P_j) \in \overline{M}_{0,|N_j|+1}$, with $1 \le j \le 4$, we can define a map*

$$\overline{\mathrm{M}}_{0,4} \longrightarrow \overline{\mathrm{M}}_{0,n}, \quad (C_0, \{Q_1, \dots, Q_4\}) \mapsto (C, \vec{p}),$$

where C is a union of C_0 and the four copies of pointed $\mathbb{P}^1$ glued by attaching the points $\{P_j\}_{j=1}^4$ to the four marked points $\{Q_j\}_{j=1}^4$. The F*-Curve $F_{N_1,\cdots,N_4}$ is the numerical equivalence class of the image of this map.*

Proposition 2.2. *Let $\mathrm{F}_{N_1,\dots,N_4}$ be an* F*-Curve on $\overline{\mathrm{M}}_{0,n}$, let X be a smooth projective homogeneous variety and suppose $\vec{\alpha}$ satisfies the codimension 1 cycle condition. Then*

$$I^{1,X}_{\beta,\vec{\alpha}} \cdot \mathrm{F}_{N_1,\dots,N_4} = \sum I^{1,X}_{\beta-\sum_{j=1}^4 \beta_j,\vec{\omega}} \prod_{j=1}^{4} I^{0,X}_{\beta_j,\alpha(N_j)\cup\omega'_j},$$

where we sum over $\vec{\omega} = \{\omega_1, \dots, \omega_4\} \in (W/W_P)^4$, and degrees $\beta = (\beta_1, \dots, \beta_4)$ such that for each $j \in \{1, \dots, 4\}$, one has that $(X, \beta_j, \alpha(N_j) \cup \omega'_j)$ satisfies the codimension 0 cycle condition.

We note the similarity of the expression in the statement Prop 2.2 with [Fak12, Prop 2.7] which gives the intersection of conformal blocks divisors $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ and F-Curves. These are equal in the case $X = \mathbb{P}^r$, $\mathfrak{g} = \mathfrak{sl}_{r+1}$, and $\ell = 1$ (see Theorem 3.1).

2.1.2. *The nonadjacent basis.* To compute classes of GW divisors in examples, we will use what is called the nonadjacent basis, which we next describe. Let G_n be a cyclic graph with n vertices labeled $S = \{1, 2, \dots, n\}$. A subset of vertices $T \subset S$ is called adjacent if $t(T)$, the number of connected components of the subgraph generated T, is 1. Since G_n is cyclic, if $t(T) = k$, then $t(T^c) = k$.

By [Car09, Proposition 1.7]. The set $B = \{\delta_T : \; t(T) \ge 2\}$ forms a basis of $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}$.

The dual of a basis element $\delta_T \in B$ is an F-curve if and only if $t(T) = 2$, and for $t(T) > 2$, dual elements are alternating sums of F-curves. In [MS15] an algorithm is given for finding a dual element.

2.1.3. *Intersecting higher codimension GW cycles with boundary classes.* For $k = n-3-c$, the locus

$$\delta^k(\overline{\mathrm{M}}_{0,n}) = \{(C,\vec{p}) \in \overline{\mathrm{M}}_{0,n} |\ C \text{ has at least } k \text{ nodes}\}$$

is effective and has dimension c. We will next give a formula for the intersection of its irreducible components with $I^{c,X}_{\beta,\vec{\alpha}}$ in case $(X,\beta,\vec{\alpha})$ satisfies the codimension c cycle condition. For the formula, we set a small amount of notation. Irreducible components of $\delta^k(\overline{\mathrm{M}}_{0,n})$ are determined by the dual graph of the curves parametrized. Such a graph is a tree with k edges, joining $k+1$ vertices, decorated by n half-edges, so that each vertex has at least 3 edges plus half-edges.

To simplify the discussion, we label the vertices $\vec{v} = \{v_1,\ldots,v_{k+1}\}$, and edges $\vec{e} = \{e_{ij}\}_{1\leq i<j\leq k+1}$, where we take e_{ij} to be zero unless v_i and v_j are connected by an edge. Half-edges are labeled $\vec{h} = \{h_j\}_{j=1}^n$, and we label the component $\delta^k(\Gamma_{\vec{v},\vec{e},\vec{h}})$.

In the formula given in Proposition 2.3, given a vertex v_i, by $\alpha(v_i)$ we mean the set of $\alpha_j \in A^*(X)$ associated to the set of half edges h_j attached to the vertex v_i. For each vertex v_i we'll also consider new classes $\gamma_{ia} \in A^*(X)$, associated to the nonzero edges e_{ia} for $i+1 < a < k+1$ and classes $\gamma^*_{ai} \in A^*(X)$, dual to $\gamma_{ai} \in A^*(X)$, associated to each nonzero edge e_{ai}. If the edge e_{ij} is zero (so vertices v_i and v_j are not connected in the dual graph), we still write down a class γ_{ij}, but it is simply not in the formula, or one can imagine that there is an edge, and, by propagation of vacua, we may assume the class is zero.

Proposition 2.3. *With notation as above, one has*

$$I^{c,X}_{\beta,\vec{\alpha}} \cdot \delta^k(\Gamma_{\vec{v},\vec{e},\vec{h}}) = \sum \prod I^{c_i,X}_{\beta_i,\alpha(v_i)\cup\{\gamma^*_{ai}\}_{a=1}^{i-1}\cup\{\gamma_{ia}\}_{a=i+1}^{k+1}},$$

where we are summing over $1 \leq i \leq k+1$ *and* $0 \leq \beta_i \leq \beta$*, such that* $\sum_{i=1}^{k+1}\beta_i = \beta$*, and taking the product over* $\gamma_{1i},\ldots,\gamma_{i-1i},\gamma_{ii+1},\ldots,\gamma_{ik+1} \in (W/W_P)^{k+1}$.

2.2. **Ingredients for the proofs of Propositions 2.2 and 2.3.**

2.2.1. *Factorization, Propagation of Vacua.* In the proof of Propositions 2.2 and 2.3 we use two properties of GW classes which we call Factorization, and Propagation of Vacua for their similarity to properties of the same name that hold for vector bundles of coinvariants and conformal blocks. Here $I^{0,X}_{\beta,\vec{\alpha}}$ plays the role of the rank of the vector bundle of co-invariants, and the $I^{1,X}_{\beta,\vec{\alpha}}$ corresponds to first Chern classes of the bundles.

To state this factorization formula [KM94, Section 2.2.6], we write the cohomology class of the diagonal for $X = G/P$: Recall from Section 1.2 that the Schubert classes X_w in $X = G/P$ are parameterized by W/W_P. For $w \in W/W_P$, let w' be the unique element so that $[X_w]\cdot[X_{w'}] = [pt] \in A^*(X)$. Then the cohomology class of the diagonal $\Delta \subset X \times X$ is

$$[\Delta] = \sum_{w\in W/W_P} X_w \otimes X_{w'} \in A^{\dim X}(X\times X). \tag{2.1}$$

Let $\gamma : \overline{\mathrm{M}}_{0,n_1+1} \times \overline{\mathrm{M}}_{0,n_2+1} \to \overline{\mathrm{M}}_{0,n_1+n_2}$ be the clutching morphism, where one attaches pointed curves by glueing them together along the last marked point for each factor. Let $\pi_i : \overline{\mathrm{M}}_{0,n_1+1} \times \overline{\mathrm{M}}_{0,n_2+1} \longrightarrow \overline{\mathrm{M}}_{0,n_i+1}$ be the projection maps.

If $(X,\beta,\{\alpha_1,\ldots,\alpha_{n_1+n_2}\})$ satisfies the codimension 1 cycle condition, then the **factorization formula** states that $\gamma^* I^{1,X}_{\beta,\{\alpha_1,\ldots,\alpha_{n_1+n_2}\}}$ decomposes as sum of divisor classes pulled back from the $\overline{\mathrm{M}}_{0,n_i+1}$ along π_i. The class pulled back from $\overline{\mathrm{M}}_{0,n_2+1}$ equals

$$\sum_{\beta_1+\beta_2=\beta, w\in W/W_P} I^{0,X}_{\beta_1,\{\alpha_1,\ldots,\alpha_{n_1},[X_w]\}} \pi_2^* I^{1,X}_{\beta_2,\{\alpha_{n_1+1},\ldots,\alpha_n,[X_{w'}]\}}$$

Note that if c_{β_1} and c_{β_2} are the corresponding codimensions in (0.2) then $c_{\beta_1} + c_{\beta_2} = c_\beta$, since the codimensions of X_w and $X_{w'}$ add up to $\dim X$.

If $(X, \beta, \{\alpha_1, \dots, \alpha_{n_1+n_2}\})$ satisfies the codimension 0 cycle condition, then $I^{0,X}_{\beta,\{\alpha_1,\dots,\alpha_{n_1+n_2}\}}$ breaks up as a sum

$$\sum_{\beta_1+\beta_2=\beta, w\in W/W_P} I^{0,X}_{\beta_1,\{\alpha_1,\dots,\alpha_{n_1},[X_w]\}}\ I^{0,X}_{\beta_2,\{\alpha_{n_1+1},\dots,\alpha_n,[X_{w'}]\}}.$$

These can be generalized to analogous factorization formulas for $I^{c,X}_{\beta,\{\alpha_1,\dots,\alpha_n,[T_0]\}}$ in case $(X, \beta, \vec{\alpha})$ satisfies the codimension c cycle condition.

The GW classes also satisfy a formula [KM94, Section 2.2.3], analogous to what is called **Propagation of Vacua** for vector bundles of conformal blocks. Namely, let $T_0 \in A^0(X)$ be the fundamental class of the space. Then if $(X, \beta, \vec{\alpha})$ satisfies the codimension c cycle condition, then

$$I^{c,X}_{\beta,\{\alpha_1,\dots,\alpha_n,[T_0]\}} = \pi^*_{n+1} I^{c,X}_{\beta,\{\alpha_1,\dots,\alpha_n\}},$$

where $\pi_{n+1} : \overline{\mathrm{M}}_{0,n+1} \to \overline{\mathrm{M}}_{0,n}$ is the projection map.

2.2.2. *Small quantum cohomology.* Assume X is a homogenous space as before. Let $T_1, \dots, T_p$ be a basis of $A^1(X)$. Let $Z[q] = Z[q_1, \dots, q_p]$ where $q_1, \dots, q_p$ are formal variables. For $\beta \in H_2(X)$, let $q^\beta = q_1^{\beta\cdot T_1} q_2^{\beta\cdot T_2} \dots q_p^{\beta\cdot T_p}$, and set $QH^*(X) = H^*(X) \otimes \mathbb{Z}[q]$. Define (see [FP97, Section 10]) a small quantum $\mathbb{Z}[q]$-algebra structure $\star$ on $QH^*(X)$ by

$$\alpha_1 \star \alpha_2 = \sum_{\gamma,\beta} q^\beta I^{0,X}_{\beta,\{\alpha_1,\alpha_2,\gamma\}} \gamma$$

here $\alpha_1, \alpha_2 \in H^*(X)$ and β runs through $H_2(X)$, and γ runs through all Schubert cycle classes.

2.2.3. *Proof of Proposition 2.2.* Let $\mathrm{F}_{N_1,\dots,N_4}$ be an F-Curve, and $I^{1,X}_{\beta,\vec{\alpha}}$ a GW-divisor on $\overline{\mathrm{M}}_{0,n}$. Without loss of generality we can rename the α_i so that $\{\alpha_i : i \in N_j\} = \{\alpha^j_j, \dots, \alpha^j_{n_j}\}$, where $n_j = |N_j|$. There is a surjective map from a product of $\overline{\mathrm{M}}_{0,4}$ and four copies of $\overline{\mathrm{M}}_{0,3}$ onto F_{N_1,N_2,N_3,N_4}. To compute the class of $I^{1,X}_{\beta,\vec{\alpha}}$, one pulls the divisor back to the product of the moduli spaces. By the factorization formula, one gets the asserted formula.

Remark 2.4. *The proof of Proposition 2.3 is analogous to the proof of Proposition 2.2.*

2.3. **Divisor intersection simplifications.** In the notation of Proposition 2.2, in order to find classes of GW divisors, one needs to know how to find, each part N_i of a partition $[n] = N_1 \cup \cdots N_4$,

$$I^{0,X}_{\beta_j,\alpha(N_j)\cup\omega'_j} \in A^0(\overline{\mathrm{M}}_{0,|N_j|+1}) \cong \mathbb{Z} \text{ and } \ I^{1,X}_{\beta-\sum_{j=1}^4,\{\omega_1,\dots,\omega_4\}} \in \mathrm{Pic}(\overline{\mathrm{M}}_{0,4}) \cong \mathbb{Z}.$$

Often these quantities can be simplified computationally. For example, setting $\alpha(N_j) \cup \omega'_j = \{\gamma_1, \dots, \gamma_k\}$,

(1) $I^{0,X}_{\beta,\{\gamma_1,\dots,\gamma_k\}}$ always reduces to 3-point GW numbers, which are the coefficients of $q^\beta[pt]$ in the small quantum product
$$[X_{\gamma_1}] \star [X_{\gamma_2}] \cdots \star [X_{\gamma_k}].$$
(2) Setting $\beta' = \beta - \sum_i \beta_i$, if for some $i \in \{1, \dots, 4\}$, the class ω_i has codimension one, then one can reduce to a three point number. For instance, say ω_4 has codimension one. Then by [FP97, Prop III, p 35],

$$I^{1,X}_{\beta',\{\omega_1,\dots,\omega_4\}} = (\omega_4 \cdot \beta')\ I^{0,X}_{\beta',\{\omega_1,\omega_2,\omega_3\}} \in \mathbb{Z} = \mathrm{Pic}(\overline{\mathrm{M}}_{0,4}). \tag{2.2}$$

(3) $I^{1,X}_{0,\{\omega_1,\dots,\omega_4\}} = 0$ and $I^{0,X}_{0,\{\gamma_1,\dots,\gamma_k\}}$ coincides with the multiplicity of the class of a point.

Simplification of the four-point numbers can often be made in terms of small quantum cohomology numbers and from identities pulled back from $\overline{\mathrm{M}}_{0,4}$.

2.3.1. The formulas to be described in this section are from [KM94, 3.2.3, Step 2]. We extend the definition of $I^{c,X}_{\beta,\vec{\alpha}}$ to allow for arbitrary $\alpha_i \in QH^*(X) = H^*(X) \otimes \mathbb{Z}[q]$ (see Section 2.2.2) by $\mathbb{Z}$-linearity (and not $\mathbb{Z}[q]$ linearity!) in α_i, and by setting

$$I^{c,X}_{\beta,\{q^{m_1}\alpha_1,\dots,q^{m_n}\alpha_n\}} = I^{c,X}_{\beta-\sum_i m_i,\{\alpha_1,\dots,\alpha_n\}}.$$

Recall that the degree $|q^{\beta}\alpha|$ is $\beta \cdot c_1(T_X) + |\alpha|$ for homogeneous $\alpha \in H^*(V)$.

Proposition 2.5. *For $\alpha_i, \alpha_j, \alpha_k, \alpha_\ell, \alpha_m \in QH^*(X)$, and homogeneous such that*

$$\sum_x |\alpha_x| = 1 + \beta \cdot c_1(T_X) + \dim X,$$

$$\begin{aligned} (2.3) \quad I^{1,X}_{\beta,\{\alpha_k,\alpha_\ell,\alpha_m,\alpha_i\star\alpha_j\}} &= I^{1,X}_{\beta,\{\alpha_j,\alpha_\ell,\alpha_m,\alpha_i\star\alpha_k\}} + I^{1,X}_{\beta,\{\alpha_i,\alpha_k,\alpha_m,\alpha_j\star\alpha_\ell\}} - I^{1,X}_{\beta,\{\alpha_i,\alpha_j,\alpha_m,\alpha_k\star\alpha_\ell\}} \\ &= I^{1,X}_{\beta,\{\alpha_j,\alpha_k,\alpha_m,\alpha_i\star\alpha_\ell\}} + I^{1,X}_{\beta,\{\alpha_i,\alpha_\ell,\alpha_m,\alpha_j\star\alpha_k\}} - I^{1,X}_{\beta,\{\alpha_i,\alpha_j,\alpha_m,\alpha_k\star\alpha_\ell\}}. \end{aligned}$$

Proof. It is easy to check that we may assume $\alpha_i, \alpha_j, \alpha_k, \alpha_\ell, \alpha_m \in H^*(X)$, by writing $\alpha_i = q^{\beta_i}\alpha'_i$ etc.

We work with the contraction morphism $\rho : \overline{\mathrm{M}}_{0,5}(X,\beta) \to \overline{\mathrm{M}}_{0,4}$. On $\overline{\mathrm{M}}_{0,4} \cong \mathbb{P}^1$, one has the divisor class identities $\delta_{ij,k\ell} = \delta_{ik,j\ell} = \delta_{i\ell,jk}$. When pulled back along ρ, these give the identities

$$\begin{aligned} (2.4) \quad &\sum_S I^{3,X}_{\beta_1,\{\alpha_i,\alpha_j,\gamma\}}\ I^{4,X}_{\beta-\beta_1,\{\alpha_k,\alpha_\ell,\alpha_m,\gamma'\}} + \sum_S I^{4,X}_{\beta_1,\{\alpha_i,\alpha_j,\alpha_m,\gamma\}}\ I^{3,X}_{\beta-\beta_1,\{\alpha_k,\alpha_\ell,\gamma'\}} \\ &= \sum_S I^{3,X}_{\beta_1,\{\alpha_i,\alpha_k,\gamma\}}\ I^{4,X}_{\beta-\beta_1,\{\alpha_j,\alpha_\ell,\alpha_m,\gamma'\}} + \sum_S I^{4,X}_{\beta_1,\{\alpha_i,\alpha_k,\alpha_m,\gamma\}}\ I^{3,X}_{\beta-\beta_1,\{\alpha_j,\alpha_\ell,\gamma'\}} \\ &= \sum_S I^{3,X}_{\beta_1,\{\alpha_i,\alpha_\ell,\gamma\}}\ I^{4,X}_{\beta-\beta_1,\{\alpha_j,\alpha_k,\alpha_m,\gamma'\}} + \sum_S I^{4,X}_{\beta_1,\{\alpha_i,\alpha_\ell,\alpha_m,\gamma\}}\ I^{3,X}_{\beta-\beta_1,\{\alpha_j,\alpha_k,\gamma'\}}, \end{aligned}$$

where $S = \{\gamma, \beta_1 \mid [\Delta] = \sum \gamma \otimes \gamma'\ \}$. Using that

$$\alpha_x \star \alpha_y = \sum_{\beta_1} q^{\beta_1} \langle \alpha_x, \alpha_y, \gamma \rangle_{\beta_1}\ \gamma',$$

one has

$$q^{\beta_1} I^{4,X}_{\beta-\beta_1,\{\alpha_a,\alpha_b,\alpha_c,\alpha_d\}} = I^{4,X}_{\beta,\{\alpha_a,\alpha_b,\alpha_c,q^{\beta_1}\alpha_d\}}.$$

We may therefore rewrite Eq 2.4 as

$$\begin{aligned} (2.5) \quad &I^{4,X}_{\beta,\{\alpha_k,\alpha_\ell,\alpha_m,\alpha_i\star\alpha_j\}} + I^{4,X}_{\beta,\{\alpha_i,\alpha_j,\alpha_m,\alpha_k\star\alpha_\ell\}} \\ &= I^{4,X}_{\beta,\{\alpha_j,\alpha_\ell,\alpha_m,\alpha_i\star\alpha_k\}} + I^{4,X}_{\beta,\{\alpha_i,\alpha_k,\alpha_m,\alpha_j\star\alpha_\ell\}} \\ &= I^{4,X}_{\beta,\{\alpha_j,\alpha_k,\alpha_m,\alpha_i\star\alpha_\ell\}} + I^{4,X}_{\beta,\{\alpha_i,\alpha_\ell,\alpha_m,\alpha_j\star\alpha_k\}}. \end{aligned}$$

□

2.3.2. *Application of Proposition 2.5.* We write a simpler version of Proposition 2.5, which when used judiciously can simplify 4-point numbers to sums of 3-point numbers.

Proposition 2.6. *For $\alpha_i, \alpha_j, \alpha_k, \alpha_\ell, \alpha_m \in QH^*(X)$, suppose that $\alpha_\ell = H$ is the class of a hyperplane, and $\alpha_m = H^{\star(t-1)}$, so that $\alpha_\ell \cdot \alpha_m = H \star H^{\star, t-1} = H^{\star,t}$. Then one can write:*

$$I^{1,X}_{\beta,\{\alpha_i,\alpha_j,\alpha_k,H^{\star t}\}} = I^{1,X}_{\beta,\{\alpha_i\star H,\alpha_k,\alpha_j,H^{\star(t-1)}\}} + I^{1,X}_{\beta,\{\alpha_i,H,\alpha_k,H^{\star(t-1)}\star\alpha_j\}} - I^{1,X}_{\beta,\{\alpha_i\star\alpha_j,\alpha_k,H,H^{\star(t-1)}\}}. \tag{2.6}$$

Remark 2.7. *If $\alpha_2, \alpha_3, \alpha_3 \in H^*(X)$ (and not in $QH^*(X)$), by* [FP97, Prop III, p 35],

$$I^{1,X}_{\beta,\{H,\alpha_2,\alpha_3,\alpha_4\}} = (H\cdot\beta) I^{0,X}_{\beta,\{\alpha_2,\alpha_3,\alpha_4\}}$$

and

$$I^{1,X}_{\beta,\{H,q^{\beta_2}\alpha_2,q^{\beta_3}\alpha_3,q^{\beta_4}\alpha_4\}} = (H\cdot\beta') I^{0,X}_{\beta',\{\alpha_2,\alpha_3,\alpha_4\}}.$$

where $\beta' = \beta - \beta_2 - \beta_3 - \beta_4$.

Therefore in Equation (2.6)*, the second and third term can be computed using small quantum cohomology. The first term has $H^{\star(t-1)}$ in the last coordinate, so the exponent in H has improved, and we can iterate the procedure to reduce to $t = 1$.*

3. Projective space

Here we prove Theorem 3.1, which links divisor classes $I^{1,\mathbb{P}^r}_{\vec{m},d}$ on $\overline{\mathrm{M}}_{0,n}$ to conformal blocks divisors for type A at level 1. In Section 3.1 we outline a potential generalization to Gromov-Witten divisors for Grassmannians $G(\ell, r+\ell)$ with higher conformal blocks divisors for type A at level ℓ. We only expect this correspondence to hold in for divisors: in Section 3.2 we give a higher codimension cycle for projective space that pushes forward to an extremal divisor, not known to be given by the first Chern class of any conformal blocks bundle.

Theorem 3.1. *Suppose we are given a pair $(r, \vec{m})$ such that $\sum_{i=1}^n m_i = (r+1)(d+1)$. Then*

$$I^{1,\mathbb{P}^r}_{\vec{m},d} \equiv c_1(\mathbb{V}(\mathrm{sl}_{r+1}, \{\omega_{m_1}, \dots, \omega_{m_n}\}, 1)).$$

Remark 3.2. *While in both cases quantum cohomology for $\mathbb{P}^r = \mathrm{Gr}(1, r+1)$ is involved, in Theorem 3.1, such classes are paired with first Chern classes of conformal blocks bundles for $\mathfrak{sl}_{r+1}$, while in Witten's Theorem they identified with ranks of conformal blocks bundles for $\mathfrak{sl}_r$.*

Proof. Conformal blocks divisors $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ are described briefly in Section 1.3. Here we are concerned with the special case when $\mathfrak{g} = \mathfrak{sl}_{r+1}$, and $\ell = 1$. In this case, for $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$, the λ_i correspond to Young diagrams with rows $\ell = 1 \geq \lambda_i^1 \geq \cdots \geq \lambda_i^r$, and the compatibility requirement is that $\sum_{i=1}^n |\lambda_i| = (r+1)(d+1)$, where $|\lambda_i| = \sum_{j=1}^r \lambda_I^j$.

It is enough to show that each divisor intersects any F-curve in the same degree. By Proposition 2.2, this amounts to proving, for any partition $N_1 \cup \cdots \cup N_4$ of $[n]$ into nonempty subsets, if we write $\vec{m}(N_j) = \{m_i : i \in N_j\}$, for any $\vec{a} = (a_1, a_2, a_3, a_4)$ with

$$\sum_{i\in N_j} m_i + a_j' = (r+1)(d_j+1), \text{ and } \sum_i a_i = (r+1)(d - \sum_i d_i + 1).$$

that $I^{0,\mathbb{P}^r}_{d_j,\vec{m}(N_j)\cup a_j'}$ is proportional to $\mathrm{rank}(\mathbb{V}(\mathrm{sl}_{r+1}, \{\omega_{m_i} : i \in N_j\} \cup \omega_{a_j'}, 1))$, and that

$$I^{1,\mathbb{P}^r}_{d-\sum_i d_i,\vec{a}} \equiv c_1(\mathbb{V}(\mathrm{sl}_{r+1}, \{\omega_{a_1}, \dots, \omega_{a_4}\}, 1)).$$

In [Fak12], Fakhruddin proved that the level one bundles in type A always have rank one. So it is enough to check:

(1) Four point classes are the same:

$$I^{1,\mathbb{P}^r}_{\beta,\vec{a}} \equiv c_1(\mathbb{V}(\mathrm{sl}_{r+1}, \{\omega_{a_1}, \dots, \omega_{a_4}\}, 1)), \text{ where } \sum_i a_i = (r+1)(\beta+1), \text{ and}$$

(2) Coefficients are the same:

$$I^{0,\mathbb{P}^r}_{d_j,\vec{m}(N_j)\cup a'_j} = \mathrm{Rank}\,\mathbb{V}(\mathfrak{sl}_r, \{\omega_{a_i} : i \in N_j\} \cup \omega_{a'_j}, 1) = 1.$$

To see that four point classes are the same: If one of the $a_i = 0$ then the class is pulled back from $\overline{\mathrm{M}}_{0,3}$, and hence zero. The conformal blocks divisor is also trivial in this case. If $\beta = 0$, then divisors from both theories are zero. Clearly $a_1 + a_2 + a_3 + a_4 \leq 4r$ and hence $\beta \in \{0, 1, 2\}$.

We show now that if $\beta = 2$, the GW divisor is zero (the same is true of the conformal blocks divisor [Fak12, Lemma 5.1]). Clearly in this case $r \geq 3$ (otherwise we will need 4 classes in $\mathbb{P}^2$ with codimensions summing to 9). We want to count maps $f : (\mathbb{P}^1, p_1, p_2, p_3, p_4) \to \mathbb{P}^r$ of degree 2 such that p_i go into specified Schubert cells (generic translates of standard cells). The image of the conic lies in a plane in $\mathbb{P}^r$. The space of such planes $\mathrm{Gr}(3, r+1)$ is of dimension $3(r-2)$. The conditions imposed on this plane are at least $\sum(a_i - 2) = 3(r+1) - 8 > 3(r-2)$, hence there are no such planes (If $a_i = 3$, then we want the $\mathbb{C}^3 \subset \mathbb{C}^{r+1}$ arising from the plane to meet a codimension 3 hyperplane in $\mathbb{C}^{r+1}$ non-trivially, which imposes one condition on the plane. Similarly if $a_i > 3$, the number of conditions imposed is $(a_i - 2)$).

Finally, if $\beta = 1$, we may assume all $a_i > 1$, because if $a_i = 1$, the GW divisor is of degree 1 (since it reduces to a three point GW number, see Section 2.3) as is the conformal blocks divisor, by [Fak12, Lemma 5.1]. Choose subspaces $L_i \subset \mathbb{C}^{r+1}$ of codimension a_i in general position. We claim the set of degree 1 maps $f : (\mathbb{P}^1, p_1, \dots, p_4) \to \mathbb{P}^r$ so that $f(p_i) \in \mathbb{P}(L_i)$ is in one-one correspondence with the set of two dimensional subspaces V of $\mathbb{C}^{r+1}$ such that $V \cap L_i \neq \{0\}$; these correspond to Schubert varieties for the partitions $(a_1 - 1, 0), (a_2 - 1, 0), (a_3 - 1, 0), (a_4 - 1, 0)$. Note that $(a_1 - 1) + (a_2 - 1) + (a_3 - 1) + (a_4 - 1) = 2(r+1) - 4 = 2(r-1)$ in the $2(r-1)$ dimensional Grassmannian $\mathrm{Gr}(2, r+1)$, so that the enumerative count in $\mathrm{Gr}(2, r+1)$ is also finite.

The correspondence takes f to the two dimensional linear subspace spanned by the image of f. The reverse correspondence takes V to the map $f : \mathbb{P}^1 \to \mathbb{P}^r$ of degree 1 with V as the span of the image of f (unique up to automorphisms of $\mathbb{P}^1$) and the points p_i are determined by the condition that $f(p_i)$ is the line $V \cap L_i$. Note that the points p_i are distinct in the reverse correspondence: By dimension counting, because if $p_1 = p_2$, then $V \cap L_1 \cap L_2$ is positive dimensional, and the numbers $a_1 - 1, a_2 - 1$ get replaced by $a_1 + a_2 - 1$ making the number of conditions larger.

Assuming $a_1 \leq a_2 \leq a_3 \leq a_4$, we want the the cardinality of the sets above to be a_1 if $a_2 + a_3 \geq a_1 + a_4$, and $r + 1 - a_4$ otherwise as is the case for the corresponding conformal blocks divisor [Fak12, Lemma 5.1]. Let $\lambda_i = a_i - 1$.

Since Littlewood-Richardson coefficients compute both the cohomology of Grassmannians $\mathrm{Gr}(2, r+1)$, and dimensions of spaces of invariants in irreducible representations of sl_2 (see e.g., [Ful00, Section 6.2]), we may compute the above cardinality as follows: Let $V(\lambda_i)$ denote the irreducible sl_2 representation corresponding to the partition λ_i above. The desired cardinality is the dimension of the space of sl_2 invariants in $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4}$. Since representations of sl_2 are self dual, this number is the number of irreducible representations (since these tensor products are multiplicity free) that occur in both $V_{\lambda_1} \otimes V_{\lambda_4}$ and $V_{\lambda_2} \otimes V_{\lambda_3}$. The tensor product of $V(\lambda_1)$ and $V(\lambda_4)$ is a multiplicity free string of representations $V(\lambda_4 - \lambda_1), V(\lambda_4 - \lambda_1 + 2), \dots, V(\lambda_1 + \lambda_4)$ (similarly for the other tensor product). Since $\lambda_4 - \lambda_1 \geq \lambda_3 - \lambda_2$, the desired intersection number is $1 + 1/2(\lambda_4 + \lambda_1 - (\lambda_4 - \lambda_1)) = a_1$ if $\lambda_4 + \lambda_1 \leq \lambda_2 + \lambda_3$, and equal to $1 + 1/2(\lambda_2 + \lambda_3 - (\lambda_4 - \lambda_1) = 1 + (r - 1 - \lambda_4) = r + 1 - a_4$ otherwise, as desired.

Since the $\vec{m}(N_j)\cup a'_j$ satisfy the $c=0$ co-cycle condition, $I^{0,\mathbb{P}^r}_{d_j,\vec{m}(N_j)\cup a'_j}$ can be computed with small quantum cohomology numbers for $\mathbb{P}^r$ and are easily seen to be 1. The ranks of the conformal blocks divisors in type A at level one are one [Fak12]. The proof of Theorem 3.1 is now complete. □

In [Gia13], it was shown for S_n-invariant divisors $c_1(\mathbb{V}(\mathfrak{sl}_{r+1},\vec{\lambda},1))$, (and then later for general divisors in [GG12]), in case $\sum_{i=1}|\lambda_i|=(r+1)(d+1)$, that the divisors $c_1(\mathbb{V}(\mathfrak{sl}_{r+1},\vec{\lambda},1))$ give maps to moduli spaces that generically parametrize configurations of weighted points that lie on a Veronese curve of degree d in $\mathbb{P}^d$. The statement of the result in Theorem 3.1 is a priori different, as it refers generically to maps of $\mathbb{P}^1$ to $\mathbb{P}^r$.

Let Γ be the set of Gromov-Witten divisors on $\overline{\mathrm{M}}_{0,n}$ coming from $X=\mathbb{P}^1$. In [Fak12], Fakhruddin proved that the set of nontrivial level one divisors $c_1(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},1))$ forms a basis for the Picard group of $\overline{\mathrm{M}}_{0,n}$. Here $\lambda_i=\omega_{a_i}$, and $\sum_{i=1}^n|\alpha_i|=\sum_i a_i=2(d+1)$. By Theorem 3.1, one also has $c_1(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},1))=I^{1,\mathbb{P}^1}_{d,\vec{a}}\in A^1(\overline{\mathrm{M}}_{0,n})$. This implies that the closed cone in $A^1(\overline{\mathrm{M}}_{0,n})$ spanned by Γ is full dimensional. Keel proved that $A^1(\overline{\mathrm{M}}_{0,n})$ generates $A^k(\overline{\mathrm{M}}_{0,n})$, and so the cone generated by products of elements of Γ, which are basepoint free cycles of codimension k, is full dimensional as well. In other words, even the simplest of these classes generate a full-dimensional subcone of strongly-basepoint free classes in the cone of effective cycles of codimension k.

3.1. **A potential generalization.** Any partition $\lambda_i=(0\le\lambda_i^1\le\cdots\lambda_i^\ell\le r)$ determines a Schubert variety X_{α_i} of dimension $|\alpha_i|=\sum_{j=1}^{\ell}\alpha_i^j$ in $\mathrm{Gr}(\ell,r+\ell)$. One could ask the following:

Question 3.3. *Are there other equivalences between conformal blocks divisors and Gromov-Witten cycles in codimension one? For instance, if for some d, one has that $r+\ell$ divides $(r+1)(d-1)$, then for $\beta=\frac{(r+1)(d-1)}{(r+\ell)}+1$, does one have*

$$I^{1,\mathrm{Gr}(\ell,r+\ell)}_{\beta,\vec{\lambda}}=c_1(\mathbb{V}(\mathfrak{sl}_{r+1},\vec{\lambda},\ell))?$$

In case $\ell=1$, one has $\beta=d$, and the answer to Question 3.3 is yes, as we saw in Theorem 3.1. In case $d=1$, one would have $\beta=1$, and the question asks whether $I^{1,\mathrm{Gr}(\ell,r+\ell)}_{1,\vec{\lambda}}=c_1(\mathbb{V}(\mathfrak{sl}_{r+1},\vec{\lambda},\ell))$. Under these assumptions, since $\sum_i|\lambda|=(r+1)(\ell+1)$, the conformal block bundles are at the critical level, and so by [BGM15], one has $c_1(\mathbb{V}(\mathfrak{sl}_{r+1},\vec{\lambda},\ell))=c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1},\vec{\lambda}^T,r))$, a positive answer to Question 3.3 in this case would imply that $I^{1,\mathrm{Gr}(\ell,r+\ell)}_{\beta,\vec{\lambda}}=I^{1,\mathrm{Gr}(r,r+\ell)}_{\beta,\vec{\lambda}^T}$, which is a tautology, since $\mathrm{Gr}(\ell,r+\ell)=\mathrm{Gr}(r,r+\ell)$.

3.2. **A higher codimension cycle that pushes forward to an extremal divisor.** We next define a divisor π_*Z on $\overline{\mathrm{M}}_{0,n-1}$, formed from the pushforward of a codimension 2 Gromov-Witten class Z on $\overline{\mathrm{M}}_{0,n}$, where we take the pushforward along the forgetful map $\overline{\mathrm{M}}_{0,n}\to\overline{\mathrm{M}}_{0,n-1}$. In case $n=7$, our extensive knowledge of $\overline{\mathrm{M}}_{0,6}$ enables us to check that π_*Z lies on a face spanned by the 2 extremal rays R_5 and R_{16}, the 5th and 16th rays on Swinarski's list of extremal rays of the nef cone [Swi11]. The ray R_{16} is not known to be spanned by a conformal blocks divisor, in spite of concerted and dedicated searches done with computer software (see Section 4.5 for a similar example).

To define the codimension 2 Gromov-Witten class on $\overline{\mathrm{M}}_{0,n}$, first suppose $\sum m_i\equiv c-1 \pmod{r+1}$. Then, $I^{c,\mathbb{P}^r}_{\vec{m},d}$ is a codimension c rationally strongly basepoint free cycle on $\overline{\mathrm{M}}_{0,n}$ with $(d+1)(r+1)+c-1=\sum m_i$. Let $\pi:\overline{\mathrm{M}}_{0,n}\to\overline{\mathrm{M}}_{o,n-c+1}$ be the map which forgets the last $c-1$ points. We wish to determine/study $\pi_*I^{c,\mathbb{P}^r}_{\vec{m},d}$ a basepoint free divisor on $\overline{\mathrm{M}}_{0,n-c+1}$.

We consider here an explicit example for $c = 2$. If $m_n = 1$, then this push forward coincides with $I^{1,\mathbb{P}^r}_{\vec{m}',d}$ on $\overline{\mathrm{M}}_{0,n-1}$ where $\vec{m}' = (m_1, \ldots, m_{n-1})$. We therefore consider an example for which $m_n > 1$, pushing forward $I^{2,\mathbb{P}^3}_{\{H_1^4,H_3^3\},2}$ from $\overline{\mathrm{M}}_{0,7}$ to $\overline{\mathrm{M}}_{0,6}$.

Let $\pi : \overline{\mathrm{M}}_{0,7} \to \overline{\mathrm{M}}_{0,6}$ be the morphism which drops the 7th marked point. For $\mathrm{Pic}(\overline{\mathrm{M}}_{0,6})$, [MS15] one nonadjacent basis is given by the set of classes

$$\{\delta_{13}, \delta_{14}, \delta_{15}, \delta_{24}, \delta_{25}, \delta_{26}, \delta_{35}, \delta_{36}, \delta_{46}, \delta_{124}, \delta_{125}, \delta_{134}, \delta_{135}, \delta_{136}, \delta_{145}, \delta_{146}\}.$$

Classes of divisors can be computed by intersecting with curves in the dual basis:

$$\begin{aligned}(3.1)\quad \{&F_{1,2,3,456}, F_{1,4,23,56}, F_{1,5,6,234}, F_{2,3,4,156}, F_{2,5,16,34}, F_{1,2,6,345},\\ &F_{3,4,5,126}, F_{3,6,12,45}, F_{4,5,6,123}, F_{3,4,12,56}, F_{5,6,12,34}, F_{1,2,34,56},\\ &(F_{5,6,13,24} + F_{1,2,3,456} + F_{2,3,4,156} - F_{2,3,16,45}), F_{2,3,16,45}, F_{1,6,23,45}, F_{4,5,16,23}\}.\end{aligned}$$

To determine the class of $\pi_*(I^{2,\mathbb{P}^3}_{\{H_1,H_2^6\},2})$ on $\overline{\mathrm{M}}_{0,6}$, we intersect $Z = I^{2,\mathbb{P}^3}_{\{H_1,H_2^6\},2}$ with the pullback $\pi^*(F)$, where F runs over the set of curves dual to the nonadjacent basis. Because of the symmetry of the Schubert classes used to define Z, we only need to keep track of where the 5th and 6th points are. The coefficients of the class in the non-adjacent basis

$$\begin{aligned}&[A, B, C, A, D, E, E, D, C, B, F, B, (F + 2A - G), G, D, D] = [1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 1, 3, 0, 1, 1]\\ &\quad = [0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0] + [1, 0, 1, 0, 1, 1, 1, 0, 2, 0, 1, 1, 2, 0, 0, 1] = R_5 + R_{16},\end{aligned}$$

where for $x \in \{p_1, p_2, p_3, p_4\}$, and $y \in \{p_5, p_6\}$ each number comes from intersecting with an F-curve:

$$\begin{aligned}&\mathrm{A} = \pi_*Z \cdot F_{\{x\}\{x\}\{x\}\{x,y,y\}} &&\mathrm{D} = \pi_*Z \cdot F_{\{x\}\{y\}\{x,y\}\{x,x\}} &&\mathrm{G} = \pi_*Z \cdot F_{\{x\}\{x\}\{x,y\}\{x,y\}}.\\ &\mathrm{B} = \pi_*Z \cdot F_{\{x\}\{x\}\{x,x\}\{y,y\}} &&\mathrm{E} = \pi_*Z \cdot F_{\{x\}\{x\}\{y\}\{x,x,y\}} &&\\ &\mathrm{C} = \pi_*Z \cdot F_{\{x\}\{y\}\{y\}\{x,x,x\}} &&\mathrm{F} = \pi_*Z \cdot F_{\{y\}\{y\}\{x,x\}\{x,x\}} &&\end{aligned}$$

We note that the divisor π_*Z contracts 12 F-Curves on $\overline{\mathrm{M}}_{0,6}$, and there are 23 extremal rays of the nef cone $\mathrm{Nef}(\overline{\mathrm{M}}_{0,6})$ that also contract those F curves. In particular, π_*Z lies on the face spanned by the 23 extremal rays. Using LRS [Avi18], one can check that as a cone, this face is 7 dimensional. In particular, a generic element on this face would be described by an effective combination of 7 divisors. In particular, since π_*Z, a sum of two extremal rays, it is a special element of this face.

4. Quadrics

In this section we demonstrate how to work with Gromov-Witten classes given by quadrics. In Propositions 4.4 and 4.5 we give criteria for Gromov-Witten divisors defined from odd quadrics to lie on extremal faces of the nef cone. As an application, in Section 4.3 we define a Gromov-Witten divisor for an odd quadric, which we prove spans an extremal ray of the nef cone on $\overline{\mathrm{M}}_{0,n}$ for any n. In Section 4.4 we define a family of extremal Gromov-Witten divisors given by even quadrics, and we show that on $\overline{\mathrm{M}}_{0,6}$, the divisor produced is not known to be given by a conformal blocks divisor.

4.1. **Background and notation.** Following [EKM08, Part 3], let $X = Q_r$ be a smooth projective quadric of even dimension $r = 2m \geq 4$ or of odd dimension $r = 2m+1 \geq 1$ given by a nondegenerate quadratic form on a vector space V, of dimension $r + 2$ over a field F, so $X \subset \mathbb{P}(V)$ (when $r = 2$, $X = \mathbb{P}^1 \times \mathbb{P}^1$, and the computations reduce to the case of $\mathbb{P}^1$, [Beh99]).

Let $H = H_1 \in A^1(X)$ the pullback of the hyperplane class in $A^1(\mathbb{P}(V))$. We let $H_i = H_1^i$ (the i-fold cup product in ordinary cohomology) for $i \in [1, r]$ and $H_0 = 1$.

The degree of the canonical bundle of a smooth projective quadric Q_r is $-r$. So for $(X, d, \vec{\alpha})$ to satisfy the codimension c cycle condition we must have that $\sum_i |\alpha_i| = c + r(d + 1)$.

In case $r = 2m+1$ is odd:

- Let W be a maximal totally isotropic subspace so $\mathbb{P}(W) \subset X$, $\dim \mathbb{P}(W) = m$. For any integer $i \in [0,m]$, let $L_i \in A_i(X)$ be the class of an i dimensional subspace of $\mathbb{P}(W)$. Then the total Chow ring of X is free with basis $\{H_i, L_i \mid i \in [0,m]\}$. Note that $H_{m+i} = 2L_{m+1-i}$ for $i \in [1, m+1]$ and $H \cdot L_i = L_{i-1}$ for any $i \in [1,m]$.
- As a basis of the rational cohomology we take $1 = H_0, H_1, \dots, H_r$.

In case $r = 2m$ is even:

- The space of maximal isotropic subspaces of V has two components. Let W_1, W_2 be representatives in each component. Now $\mathbb{P}(W_a) \subset X$, $a = 1,2$ and let $\xi_1, \xi_2 \in A_m(X)$ be their cycle classes. For $i \in [0, m-1]$, let $L_i \in A_i(X)$ be the cycle class of an i dimensional subspace of $\mathbb{P}(W_1)$ (note that we get the same cycle class if W_1 is replaced by W_2 here). The total Chow ring of X is free with basis $H_0 = 1$, H_1, …,H_{m-1}, ξ_1, ξ_2, L_{m-1} …, L_0. We also have $H \cdot L_i = L_{i-1}$ for any $i \in [1, m-1]$, $H_m = \xi_1 + \xi_2$ and $H \cdot \xi_a = L_{m-1}$ for $a = 1,2$, so that $H_{m+1} = 2L_{m-1}$.
- For even dimensional quadrics, as a basis of the rational cohomology we take

$$1 = H_0, H_1, \dots, H_{m-1}, \xi_1, \xi_2, H_{m+1}, \dots, H_r.$$

In both cases, for $X = Q_r$ (even or odd), $H^{\star j} = H_j$ if $j < r$. If $j = r$, then $H^{\star j}$ equals H_j plus a multiple of q times the identity in cohomology. But a four point number with one of the terms equalling identity in cohomology is zero. Therefore we may always write $< H_j, y_1, y_2, y_3 >_\beta = < H^{\star, j}, y_1, y_2, y_3 >_\beta$ and apply Proposition 2.6 to simplify intersection formulas.

The cohomology of an even quadric is generated by the hyperplane class except in the middle dimension. But we cannot have a 4 point number with all four terms in the middle dimension, since the codimensions need to add up to 1 mod r. Therefore 4 point numbers for even quadrics are computable with these methods.

To compute classes, we determine certain facts about the quantum cohomology of $X = Q_r$.

Lemma 4.1. $I^{0,Q_r}_{1,\{H_1,H_r,H_{r-1}\}} = 4$.

Lemma 4.2.

$$H_i \star H_j = \begin{cases} H_{i+j} & \textit{if } i+j < r; \\ H_r + 2qH_0 & \textit{if } i+j = r; \\ 4qH_\ell & \textit{if } i+j = r+\ell, \textit{ with } i < r \textit{ and } j < r; \\ 2qH_i & \textit{if } i < r, \textit{ and } j = r; \\ 4q^2 H_0 & \textit{if } i = j = r. \end{cases}$$

Remark 4.3. *The formulas in Lemmas 4.1 and 4.2 hold for r both even and odd. Formulas specific to the even case are given in Section 4.4.*

Proof. (of Lemma 4.1) For the first assertion, we need to count lines in the quadric Q which pass through a point P, and a fixed line L in the quadric. Clearly four times this count gives us the desired answer since H_r and H_{r-1} are twice the classes of a point and a line respectively. Consider the projective space spanned by the point P and the fixed line L, a $\mathbb{P}^2$. The quadric, restricted to this $\mathbb{P}^2$ splits as a product $Q = LL'$ since it contains L, we may assume that L' passes through the point P ($P \notin L$), and L' is the unique line we are looking for (it certainly meets L). ∎

Proof. (of Lemma 4.2) For odd quadrics, one can show that $H^{\star i} = H_i$ if $i \leq r-1$, and $H^{\star r} = H \star H_{r-1} = H_r + 2q \cdot 1$, since the dual of 1 is $\frac{1}{2}H_r$. $H^{\star r+1} = 2qH + H \star H_r = 4qH$, since the dual of H is $\frac{1}{2}H_{r-1}$. For even quadrics we need the action of orthogonal group on the space of

$m+1$ isotropic subspaces of $\mathbb{C}^{2m+2}$ has two components. The dimension of intersection of two subspaces in the same connected component is constant modulo two. Since the three point number $\langle H, H_r, H_{r-1}\rangle_1$ is equal to 4, the multiplication rules for $H_i \star H_j$ are the same.

□

4.2. **Extremality results.** As the following results show, it is straightforward to design divisors $I^{1,Q_r}_{d,\vec{a}}$, for $Q_r \subset \mathbb{P}^{r+1}$ an odd quadric, that lie on extremal faces of the nef cone.

Proposition 4.4. *Let $Q_r \subset \mathbb{P}^{r+1}$ be an odd quadric, and $I^{1,Q_r}_{d,\vec{a}}$ a GW-divisor. If there is an index $i \in [n]$ such that $a_i = r$ and $J \subset [n] \setminus i$, such that for all $j \in J$, $1 \le a_j \le r$, and $\sum_{j\in J} a_j = r$, then $I^{1,Q_r}_{d,\vec{a}}$ contracts any F-curve of the form $F_{I,A,B,C}$, for $I = J \cup \{i\}$, and lies on a face of the nef cone.*

Proof. Let $r = 2m+1$, and two indices i and $j \in [n]$ with $a_i = a_j = r$, then the divisor $I^{1,Q_r}_{d,\vec{a}}$ will kill any F-curve of the form $F_{I,A,B,C}$ where $I = \{a_i, a_j\}$, since $H_r \star H_r = 4qH_0$, and $4qI^{1,Q_r}_{d,\{H_0,\alpha_1,\alpha_2,\alpha_3\}} = 0$ for all possible $\alpha_1, \alpha_2, \alpha_3$ under consideration. More generally if there is an index $i \in [n]$ such that $a_i = H_r$, and $J \subset [n] \setminus i$, such that for all $j \in J$, $1 \le a_j \le r$, and $\sum_{j\in J} a_j = r$, one has $a_i = H_i$, and the star product of classes in J is H_r, so the star product of classes in $I = J \cup \{i\}$ is $H_r \star H_r = 4qH_0$, and the result follows. □

Proposition 4.5. *Let $Q_r \subset \mathbb{P}^{r+1}$ be an odd quadric, and $I^{1,Q_r}_{d,\vec{a}}$ a GW-divisor with $d \le 4$.*

- $d = 1$ *If there are indices a_1 and a_2 such that $a_1 + a_2 > r$, then $I^{1,Q_r}_{1,\vec{a}}$ will kill any F-curve of the form $F_{A,B,C,D}$ where $\{a_1, a_2\} \subset A$.*
- $d = 2$ *If there are indices a_1, a_2, b_1, and b_2 such that $a_1 + a_2 > r$, and $b_1 + b_2 > r$ then $I^{1,Q_r}_{2,\vec{a}}$ will kill any F-curve of the form $F_{A,B,C,D}$ where $\{a_1, a_2\} \subset A$ and $\{b_1, b_2\} \subset B$.*
- $d = 3$ *If there are indices a_1, a_2, b_1, b_2, c_1, and c_2, such that $a_1+a_2 > r$, $b_1+b_2 > r$ and $c_1+c_2 > r$ then $I^{1,Q_r}_{3,\vec{a}}$ will kill any F-curve of the form $F_{A,B,C,D}$ where $\{a_1, a_2\} \subset A$, $\{b_1, b_2\} \subset B$, and $\{c_1, c_2\} \subset C$.*
- $d = 4$ *If there are indices a_1, a_2, b_1, b_2, c_1, c_2, d_1, and d_2, such that $a_1 + a_2 > r$, $b_1 + b_2 > r$, $c_1 + c_2 > r$, and $d_1 + d_2 > r$ then $I^{1,Q_r}_{4,\vec{a}}$ will kill any F-curve of the form $F_{A,B,C,D}$ where $\{a_1, a_2\} \subset A$, $\{b_1, b_2\} \subset B$, $\{c_1, c_2\} \subset C$, and $\{d_1, d_2\} \subset C$.*

Proof. Intersections on the leg bring the degree down by one, leaving the spine at degree zero. □

4.3. **A Gromov-Witten class that spans an extremal ray of** $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$**.** Consider the Gromov-Witten divisor $I^{1,Q_r}_{n-2,\{1,r^{n-1}\}}$ on $\overline{\mathrm{M}}_{0,n}$, where $n = 2r+2$, then by Proposition 4.4, one has that $I^{1,Q_r}_{n-2,\{1,r^{n-1}\}}$ contracts all F-curves of the form

$$\{F_{1,1,i,j} : 1 \le i \le r-1, i+j+2 = n\},\ n = 2r+2.$$

Indeed, while the i parameter is small, the $j \ge r+1$ in case. The class is not trivial as it is nonzero on the curve $F_{1,1,r,r}$. As was shown in [Fak12, Proposition 5.2], $c_1(\mathfrak{sl}_n, \omega_1^n, 1)$ also intersects $F_{1,1,i,j}$, where $1 \le i \le r-1$ in degree zero, and $F_{1,1,r,r}$ in nonzero degree, $I^{1,Q_r}_{n-2,\{1,r^{n-1}\}}$ is proportional to $c_1(\mathfrak{sl}_n, \omega_1^n, 1)$. Since this family of curves is independent, we conclude that $I^{1,Q_r}_{n-2,\{1,r^{n-1}\}}$ spans an extremal ray of the nef cone. Incidentally, by [BGM15, Proposition 1.6], one has that $c_1(\mathfrak{sl}_n, \omega_1^n, 1) = c_1(\mathfrak{sl}_2, \omega_1^n, n-1)$.

For a specific example, one can directly calculate, using the nonadjacent basis (see [MS15]), that

$$\frac{1}{16} I^{1,Q_5}_{4,\{1,r^5\}} = \delta_{13} + \delta_{15} + \delta_{24} + \delta_{26} + \delta_{35} + \delta_{46} + 2\delta_{135} = R_1,$$

where R_1 is the first ray on the list of extremal rays of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,6})$ listed in [Swi11].

4.4. **Even Quadrics.** In Section 4.5 we give a Gromov-Witten divisor for an even quadric Q_4 that is not known to be spanned by a conformal blocks divisor.

Because of the cohomology class in the middle dimension, the classes for the even quadrics $X = Q_{2m}$ can be different, depending on whether m is even or odd. Moreover, when $m = 2$, and $m = 3$, differences in the symmetry causes the classes to behave differently than in the general case. To compute classes, the following facts are used.

Lemma 4.6. (1) *$H \star \xi_1 = H \star \xi_2 = \frac{1}{2}H_{m+1}$ (for degree reasons there are no q terms).*
(2) *$H_m \star H_m = H_{2m} + 2q \cdot H_0$.*
(3) *If m is odd, then $\langle \xi_1, \xi_2, [pt]\rangle_1 = 0$, and so $\langle \xi_1, \xi_1, [pt]\rangle_1 = 1$. Therefore $\xi_1 \star \xi_2 = [pt]$ and $\xi_1 \star \xi_1 = \xi_2 \star \xi_2 = q \cdot 1$*
(4) *If m is even then $\langle \xi_1, \xi_1, [pt]\rangle_1 = 0$, and hence $\langle \xi_1, \xi_2, [pt]\rangle_1 = 1$. Therefore $\xi_1 \star \xi_2 = q \cdot 1$ and $\xi_1 \star \xi_1 = \xi_2 \star \xi_2 = [pt]$*

Proof. We need to compute $\xi_1 \star \xi_2$ and $\xi_1 \star \xi_1 = \xi_2 \star \xi_2$. But $H_m \star H_m = (\xi_1 + \xi_2) \star (\xi_1 + \xi_2) = 2\xi_1 \star \xi_2 + (\xi_1 \star \xi_1 + \xi_2 \star \xi_2)$, Therefore the $q \cdot 1$ terms in $\xi_1 \star \xi_2$ and $\xi_1 \star \xi_1$ add to 1, so one of them should be one and the other 0. In the second case (the first is similar) pick linear spaces M and M' in the quadric Q_r in general position and with cohomology class ξ_1. We get linear spaces $M, M' \subseteq \mathbb{C}^{2m+2}$ of dimension $m + 1$ each. The dimension of intersection of M and M' is congruent modulo two to $m + 1$, an odd number. Therefore we may assume $M \cap M'$ is one dimensional. Now we want to count lines L in the quadric through M, M' and a general point A in Q_r. Consider the span of A and M giving us a $P = \mathbb{P}^{m+1}$ in $\mathbb{P}^{2m+1}$. The quadric restricted to P equals MT, T a hyperplane in P which contains A. $M' \cap P$ is entirely contained in M, and we may assume that it does not intersect $T \cap M$. The line L has to stay in T, and pass through $M' \cap P$ which does not intersect T. This is not possible. □

4.5. **An extremal Gromov-Witten class from an even quadric.** Using intersections as before, one can show that in the standard nonadjacent basis, for $X = Q_4$

$$(4.1)\quad \begin{aligned} I^{1,X}_{2,\{H_1,\xi_1,\xi_1,\xi_1,\xi_2,H_4\}} &= [0,1,1,0,2,0,2,0,2,1,2,1,2,0,0,2] \\ &= [0,1,0,0,1,0,1,0,1,1,1,1,1,0,0,1] + [0,0,1,0,1,0,1,0,1,0,1,0,1,0,0,1] = R_{20} + R_3. \end{aligned}$$

The ray R_3 is known to be spanned by any conformal blocks divisor of the form

$$\mathrm{R}_3 = \rho\ c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^3, \ell\omega_1, \omega_{r-2}, \ell\omega_r\}, \ell)),$$

where $r \geq 3$, $\ell \geq 1$, for some positive rational ρ. But R_{20} is not known to be spanned by any conformal blocks divisor.

5. Two related questions

5.1. **Cycles from Gromov-Witten theory of Blow-ups.** Let X be a convex variety (e.g., a homogenous projective variety) of dimension m, and $\pi : \widetilde{X} \to X$ the blow up of X at a point $P \in X$. There is a natural inclusion $\pi^* : A^*(X) \to A^*(\widetilde{X})$ via pull-back of cycles. Note that $\pi_* \circ \pi^*$ is the identity on $A^*(X)$ (here $\pi_* : A^*(\widetilde{X}) \to A^*(X)$ is the natural push forward map on cycles).

It follows from [Gat01, Lemma 2.2] that if $\vec{\alpha}$ is an n-tuple of effective cycles in $A^*(X)$, and $\beta \in A_1(X)$, as we write in case the codimension c cycle condition is satisfied by the triple $(X, \beta, \vec{\alpha})$,

$$(5.1)\qquad I^{c,X}_{\beta,\{\alpha_1,\ldots,\alpha_n\}} = I^{c,\widetilde{X}}_{\pi^*\beta,\{\pi^*\alpha_1,\ldots,\pi^*\alpha_n\}}.$$

Now let $X=\mathbb{P}^r$. The cohomology of X is generated by cycle classes of linear subspaces $L_d\subset\mathbb{P}^r$ of some codimension d. The cycle classes of these linear spaces are the same, denoted $[L_d]$. Now choose one such linear subspace $L_d\subset\mathbb{P}^r$ of codimension d which passes through P, then $\pi^*[L_d]=L'_d+T_d$ where $L'_d\subset\widetilde{X}$ is the strict transform of L_d, and T_d is the class of a $\dim(L_d)=r-d$ linear subspace of the exceptional divisor, on $\widetilde{X}$. Therefore if α_i are cycle classes of positive dimensional subspaces L_{a_i} in $\mathbb{P}^r$ of codimension a_i (so no point classes), then one can rewrite Equation (5.1) as follows,

$$I^{c,X}_{\beta,\{\alpha_1,\dots,\alpha_n\}}=I^{c,\widetilde{X}}_{\pi^*\beta,\{\pi^*\alpha_1,\dots,\pi^*\alpha_n\}}=I^{c,\widetilde{X}}_{\pi^*\beta,\{L'_{a_i}+T_{a_i}\}_{i=1}^n}=\sum_{S\subset\{1,\dots,n\}}I^{c,\widetilde{X}}_{\pi^*\beta,\{L'_{a_i}\}_{i\in S}\cup\{T_{a_i}\}_{i\in S^c}}.\tag{5.2}$$

We have therefore decomposed the Gromov-Witten classes into a sum of (possibly non effective) cycles on $\overline{\mathrm{M}}_{0,n}$ by expanding the above quantity (5.2).

5.2. **Fedorchuk's divisors.** For $\sum_i a_i=(r+1)(d+1)$, recall we have shown

$$I^{1,\mathbb{P}^r}_{d,\vec{a}}\equiv c_1(\mathbb{V}(\mathrm{sl}_{r+1},\{\omega_{a_1},\dots,\omega_{a_n}\},1)).\tag{5.3}$$

Assuming that none of the a_i are zero, we consider the divisor

$$\mathbb{D}'=2c_1(\mathbb{V}(\mathrm{sl}_{r+1},\{\omega_{a_1},\dots,\omega_{a_n}\},1))-\sum_{r+1|\sum_{i\in I}a_i}\Delta_{I,J},$$

which Fedorchuk [Fed14, Equation (7.0.17)] has proved is nef, and an effective sum of boundary classes. However $\mathbb{D}'$ is not known to be semi-ample (i.e., that some multiple is basepoint free). Using [GG12, Proposition 1.3] $2c_1(\mathbb{V}(\mathrm{sl}_{r+1},\{\omega_{a_1},\dots,\omega_{a_n}\},1))=c_1(\mathbb{V}(\mathrm{sl}_{2r+2},\{\omega_{2a_1},\dots,\omega_{2a_n}\},1))$, we can rewrite the expression with Fedorchuk's divisor as (with $\vec{m}=2\vec{a}$), $I^{1,\mathbb{P}^{2r+1}}_{d,\vec{m}}=\mathbb{D}'+\sum_{r+1|\sum_{i\in S}a_i}\Delta_{S,S^c}$. Can $\mathbb{D}'$ be characterized by Gromov-Witten theory of blow-ups, for example, is $\mathbb{D}'$ equivalent to some combination of terms in the following natural decomposition (use (5.2) for $X=\mathbb{P}^{2r+1}$)?

$$I^{1,\mathbb{P}^{2r+1}}_{d,2\vec{m}}=\sum_{S\subsetneq\{1,\dots,n\},}I^{1,\widetilde{X}}_{\pi^*\beta,\{L'_{m_i}\}_{i\in S}\cup\{T_{m_i}\}_{i\in S^c}}.\tag{5.4}$$

5.3. **Divisors from the Gromov-Witten theory of pairs.** Consider the case of a projective space $X=\mathbb{P}^r$ and a hyperplane H in X. Let $s>1$ and $\alpha=(\alpha_1,\dots,\alpha_s)$ be an s-tuple of positive integers such that $\sum_{i=1}^s\alpha_i=d$. Define the space $\overline{M}_{0,n,s}(X,d\mid\alpha)=\overline{M}_{0,n,s}(H/X,d\mid\alpha)$ to be the closure in $\overline{M}_{0,n+s}(X,d)$ of the set of irreducible stable maps $(C,x_1,\dots,x_n,y_1,\dots,y_s,f)$ of degree d to X with $f(C)\not\subset H$ such that the divisor f^*H on $C\cong\mathbb{P}^1$ is equal to $\sum_i\alpha_iy_i$ (equality of cycles, not just linear equivalence). This implies $f(y_i)\in H$ (since α_i are assumed to be positive).

Vakil [Vak00] has shown that each irreducible component of $\overline{M}_{0,n,s}(X,d\mid\alpha)$ has the expected dimension, which is equal to $\dim\overline{M}_{0,n}(X,d)-\sum_{i=1}^s(\alpha_i-1)$. Let $\gamma_1,\dots,\gamma_n\in A^*(X)$ and $\mu_1,\dots,\mu_s\in A^*(H)$ and set $\sum_i\operatorname{codim}\gamma_j+\sum_j\operatorname{codim}\mu_i=\tau$. Then one can form the cycle

$$(\mathrm{ev}^*_{x_1}\gamma_1\dots\mathrm{ev}^*_{x_n}\gamma_n)\cdot(\mathrm{ev}^*_{y_1}\mu_1\dots\mathrm{ev}^*_{x_s}\mu_s)\cap[\overline{M}_{0,n,s}(X,d\mid\alpha)]\in A^*(\overline{M}_{0,n,s}(X,d\mid\alpha)),$$

which has homological degree $\dim\overline{M}_{0,n,s}(X,d\mid\alpha)-\tau$, pushforward to $\overline{M}_{0,n+s}$ the same degree, and is a class of codimension c if $\dim\overline{M}_{0,n,s}(X,d\mid\alpha)-\tau=\dim\overline{M}_{0,n+s}-c$, which simplifies to

$$d(r+1)+r+c=\sum\alpha_i+\sum\gamma_j+\sum\mu_i.$$

Let $I^{c,H/X}_{d,\alpha}(\gamma_1\otimes\dots\otimes\gamma_n\mid\mu_1\otimes\dots\otimes\mu_s)\in A^c(\overline{M}_{0,n+s})$ denote the push-forward cycle. It is easy to see that it is effective (by Kleiman's theorem). However, it is not clear that $I^{c,H/X}_{d,\alpha}(\gamma_1\otimes\dots\otimes\gamma_n\mid$

$\mu_1 \otimes \ldots \otimes \mu_s)$ is basepoint free. To prove so using our methods so far, one would need to know the dimension of fibers of $\overline{M}_{0,n,s}(H/X, d \mid \alpha) \to \overline{M}_{0,n+s}$, or show this map is flat.

Remark 5.1. *Loci of enumerative significance inside $(G/B)^n$ were used in recent work of the first author and J. Kiers* [Bel19, BK18] *to determine the extremal rays of the $\mathbb{Q}$-cone of G-invariant effective divisors on $(G/B)^n$, see* [BK18, Theorem 1.6]. *These loci bear a resemblance to the Gromov-Witten loci considered in this paper, in that we are varying the marked curve and keeping the point in $(G/B)^n$ fixed here: In* [BK18] *one considers loci of points $\vec{g} \in (G/B)^n$ such that there exist points of G/P which satisfy enumerative constraints given by $\vec{g}$. A point in G/P can be viewed as a degree zero map from a fixed n-marked genus zero curve to G/P. Maps of non-zero degrees are considered in the multiplicative/quantum generalizations of this problem. The Gromov-Witten loci are basepoint free, whereas in* [Bel19, BK18]*, the loci obtained (under some enumerative assumptions) are strongly rigid* [BK18, Theorem 1.6, (b)]. *It is perhaps fruitful to look at "universal" GW enumerative loci in $\overline{\mathrm{M}}_{0,n} \times (G/B)^n$, but we have not pursued this here.*

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