# CONFORMAL BLOCKS FROM VERTEX ALGEBRAS AND THEIR CONNECTIONS ON $\overline{\mathcal{M}}_{g,n}$

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ABSTRACT. We show that coinvariants of modules over conformal vertex algebras give rise to quasi-coherent sheaves on moduli of stable pointed curves. These generalize Verlinde bundles or vector bundles of conformal blocks defined using affine Lie algebras studied first by Tsuchiya–Kanie, Tsuchiya–Ueno–Yamada, and extend work of a number of researchers. The sheaves carry a twisted logarithmic $\mathcal{D}$-module structure, and hence support a projectively flat connection. We identify the logarithmic Atiyah algebra acting on them, generalizing work of Tsuchimoto for affine Lie algebras.

Vertex algebras, whose roots stretch back to the revolutionary ideas of monstrous moonshine and string theory, have been described as analogues of commutative associative algebras and complex Lie algebras. They extend elemental structural concepts in the representation theory of affine Kac-Moody algebras, conformal field theory, finite group theory and combinatorics, integrable systems, and modular functions [39, 24, 37, 28, 18, 35, 63, 58, 47, 45].

We study geometric realizations of representations of a conformal vertex algebra $V$. The idea, originating in [66, 67], formulated here from the perspective of [18, 35], is to assign a $V$-module $M^i$ to each marked point $P_i$ on an algebraic curve $C$, and then quotient by the action of a Lie algebra (§4.1) on $\otimes_i M^i$ which reflects the geometry of the pointed curve $(C, P_\bullet)$. We do this in families to form a sheaf $\mathbb{V}(V; M^\bullet)$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves (Def. 3). This generalizes prior work on families of smooth pointed curves [35, 18], coinvariants obtained from affine Lie algebras [66, 67, 65], and special cases using vertex algebras [68, 60, 5]. Our main result is:

**Theorem.** *Given $n$ simple modules $M^i$ of conformal dimension $a_i$ over a conformal vertex algebra $V$ of central charge $c$, the logarithmic Atiyah algebra $\frac{c}{2}\mathcal{A}_\Lambda + \sum_{i=1}^n a_i \mathcal{A}_{\Psi_i}$ acts on the quasi-coherent sheaf of coinvariants $\mathbb{V}(V; M^\bullet)$ specifying a twisted logarithmic $\mathcal{D}$-module structure.*

The Theorem is proved in §7 and a more precise statement with more details is given there; logarithmic Atiyah algebras are reviewed in §6.1. The analogue of the Theorem for sheaves of coinvariants of integrable representations of an affine Kac-Moody algebra was proved in [65].

As an application, when $\mathbb{V}(V; M^\bullet)$ is the sheaf of sections of a vector bundle on $\mathcal{M}_{g,n}$ (this is known to be verified in some cases, see §8.1), the action of the Atiyah algebra from the Theorem gives the Chern character of $\mathbb{V}(V; M^\bullet)$ on $\mathcal{M}_{g,n}$:

**Corollary.** *When $\mathbb{V}(V; M^\bullet)$ is locally free of finite rank on $\mathcal{M}_{g,n}$ and $c, a_i \in \mathbb{Q}$, one has*

$$\mathrm{ch}\left(\mathbb{V}(V; M^\bullet)\right) = \mathrm{rank}\, \mathbb{V}(V; M^\bullet) \cdot \exp\left(\frac{c}{2}\lambda + \sum_{i=1}^n a_i \psi_i\right) \in H^*(\mathcal{M}_{g,n}, \mathbb{Q}).$$

*In particular, Chern classes of $\mathbb{V}(V; M^\bullet)$ lie in the tautological ring.*

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The formula in the Corollary, proved in §8, extends the computation of the Chern classes of vector bundles of coinvariants on $\mathcal{M}_{g,n}$ in the affine Lie algebra case from [56].

0.1. **Overview.** Here we give an extension of the construction of sheaves of coinvariants $\mathbb{V}(V;M^\bullet)$ of modules over vertex algebras on the locus of smooth curves $\mathcal{M}_{g,n}$ from [35], to stable pointed curves $\overline{\mathcal{M}}_{g,n}$ (see §5.3.2). The problem of defining such bundles on $\overline{\mathcal{M}}_{g,n}$ has been studied before in particular cases, the most well known being for affine Kac-Moody algebras, but also for vertex algebras in more generality (see §5.3.3 for an account). We note that *conformal blocks* are defined as dual to coinvariants. In the case of vertex algebras arising from Kac-Moody algebras, conformal blocks are known to be vector spaces canonically isomorphic to generalized theta functions [16, 33, 51, 61, 17, 47].

We work with modules over a broad class of vertex algebras, called *conformal vertex algebras*, which admit an action of the Virasoro algebra. We define these and related objects in §1. Vertex algebras and their modules depend on a formal variable $z$. The geometric realization from [35] starts by considering $z$ as the formal coordinate at a point on an algebraic curve. One is thus led to consider the moduli space $\widehat{\overline{\mathcal{M}}}_{g,n}$ parametrizing objects $(C, P_\bullet = (P_1, \ldots, P_n), t_\bullet = (t_1, \ldots, t_n))$, where $(C, P_\bullet)$ is a stable $n$-pointed curve of genus $g$, and $t_i$ is a formal coordinate at $P_i$ (§2).

The strategy is to first define sheaves of coinvariants $\widehat{\overline{\mathbb{V}}}(V;M^\bullet)$ on $\widehat{\overline{\mathcal{M}}}_{g,n}$ (Def. 1). Then one shows that $\widehat{\overline{\mathbb{V}}}(V;M^\bullet)$ descends to a sheaf $\mathbb{V}^J(V;M^\bullet)$ on $J = \overline{\mathcal{J}}^{1,\times}_{g,n}$, the moduli space parametrizing points $(C, P_\bullet, \tau_\bullet)$, where $\tau_i$ is a non-zero 1-jet at $P_i$ (Def. 2). A second descent allows one to define $\mathbb{V}(V;M^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ (Def. 3). Fibers of the sheaves are canonically isomorphic to the vector spaces of coinvariants (21) and (22). Figure 5 depicts relationships between these sheaves and spaces.

The action of the Virasoro algebra on the vertex algebra modules is responsible for the presence of the twisted logarithmic $\mathcal{D}$-module structure on sheaves of coinvariants. The twisted $\mathcal{D}$-module structure of sheaves of coinvariants on families of smooth curves has been presented in [35] as an integral part of their construction. Twisted logarithmic $\mathcal{D}$-module structures on sheaves over a smooth scheme $S$ are parametrized by elements in the $\mathbb{C}$-vector space $\mathrm{Ext}^1(\mathcal{T}_S(-\log\Delta), \mathcal{O}_S)$, that is, the space of logarithmic Atiyah algebras [22]. The merit of the Theorem is to identify the logarithmic Atiyah algebra acting on sheaves of coinvariants.

Crucial to this identification is the *Virasoro uniformization*, Theorem 2.1, which gives a Lie-theoretic realization of the logarithmic tangent bundle on families of stable curves (after [13], [22], [50], [67]). This was proved in [67]; we give an alternative proof of this result in §2, extending to families of stable curves with singularities the argument for families of smooth curves given in [35].

Throughout, we work over the smooth DM stack of stable curves $\overline{\mathcal{M}}_{g,n}$, in particular we assume $2g - 2 + n > 0$.

0.2. **Future directions.** The results of this paper will serve as a cornerstone, for the following future developments. In the forthcoming [26], we investigate the factorization property of spaces of coinvariants on singular curves, here defined. In the case of finite rank, following the affine Lie algebra case treated in [57], the factorization property will allow to determine the Chern classes of the sheaves $\mathbb{V}(V;M^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ from the Corollary by applying the same recursion solved in [57].

Classes constructed from affine Lie algebras are known to be semi-ample in genus zero [30]; shown to determine full dimensional subcones of nef cones in all codimension, and used to produce new birational models of moduli of curves (e.g., [12], [42], [43], [8]). We note that while the extension to the boundary of the sheaves we consider here could have been done with the technology in place when [35] was written, the motivation to do so comes from these recent results. As the representation theory of conformal vertex algebras is richer than for affine Lie algebras, sheaves $\mathbb{V}(V;M^\bullet)$ are expected to provide new information about the birational geometry of $\overline{\mathcal{M}}_{g,n}$.

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## 1. Background

After reviewing the Virasoro algebra in §1.1, we define a conformal vertex algebra $V$, in §1.2, and modules over $V$, in §1.3. Vertex algebras and their modules depend on a formal variable $z$. In §3 we will consider geometric realizations of vertex algebras and their modules independent of the variable $z$. For this, we employ certain Lie groups and Lie algebras related to automorphisms and derivations of $\mathbb{C}[\![z]\!]$ and $\mathbb{C}(\!(z)\!)$ reviewed in §1.4. The Lie algebras arise as Lie sub- or quotient algebras of the Virasoro algebra. In §1.6–§1.7 we define the particularly important Lie group $\mathrm{Aut}_+\mathcal{O}$ and Lie algebra $\mathfrak{L}(V)$, emphasizing certain properties of their actions on $V$-modules.

1.1. **The Virasoro algebra.** Given a $\mathbb{C}$-algebra $R$, consider the Lie algebra $\mathrm{Der}\,\mathcal{K}(R) = R(\!(z)\!)\partial_z$ generated over $R$ by the derivations $L_p := -z^{p+1}\partial_z$, for $p \in \mathbb{Z}$, with relations $[L_p, L_q] = (p-q)L_{p+q}$. Let $\mathfrak{gl}_1$ be the functor which assigns to $R$ the Lie algebra $R$ with the trivial Lie bracket. The *Virasoro algebra* Vir is the functor of Lie algebras defined as the central extension

$$0 \to \mathfrak{gl}_1 \cdot K \to \mathrm{Vir} \to \mathrm{Der}\,\mathcal{K} \to 0$$

with bracket

$$[K, L_p] = 0, \qquad [L_p, L_q] = (p-q)L_{p+q} + \frac{K}{12}(p^3 - p)\delta_{p+q,0}.$$

1.2. **Conformal vertex algebras.** A *conformal vertex algebra* is the datum $(V, |0\rangle, \omega, Y(\cdot, z))$, where:

(i) $V = \oplus_{i\geq 0} V_i$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$–vector space with $\dim V_i < \infty$;
(ii) $|0\rangle$ is an element in $V_0$, called the *vacuum vector*;
(iii) $\omega$ is an element in $V_2$, called the *conformal vector*;
(iv) $Y(\cdot, z)$ is a linear operation

$$\begin{aligned} Y(\cdot, z)\colon V &\to \mathrm{End}(V)[\![z, z^{-1}]\!] \\ A &\mapsto Y(A, z) := \sum_{i\in\mathbb{Z}} A_{(i)} z^{-i-1}. \end{aligned}$$

The series $Y(A, z)$ is called the *vertex operator* assigned to $A$.

The datum $(V, |0\rangle, \omega, Y(\cdot, z))$ is required to satisfy the following axioms:

(a) *(vertex operators are fields)* for all $A, B \in V$, one has $A_{(i)}B = 0$, for $i >> 0$;
(b) *(vertex operators and the vacuum)* one has $Y(|0\rangle, z) = \mathrm{id}_V$, that is:

$$|0\rangle_{(-1)} = \mathrm{id}_V \qquad \text{and} \qquad |0\rangle_{(i)} = 0, \quad \text{for } i \neq -1,$$

and for all $A \in V$, one has $Y(A, z)|0\rangle \in A + zV[\![z]\!]$, that is:

$$A_{(-1)}|0\rangle = A \qquad \text{and} \qquad A_{(i)}|0\rangle = 0, \qquad \text{for } i \geq 0;$$

(c) *(weak commutativity)* for all $A, B \in V$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^N\,[Y(A, z), Y(B, w)] = 0 \quad \text{in } \mathrm{End}(V)[\![z^{\pm 1}, w^{\pm 1}]\!];$$

(d) *(conformal structure)* the Fourier coefficients of the vertex operator $Y(\omega, z)$ satisfy the Virasoro relations:

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p-q)\,\omega_{(p+q+1)} + \frac{c}{12}\,\delta_{p+q,0}\,(p^3-p)\,\mathrm{id}_V,$$

for some constant $c \in \mathbb{C}$, called the *central charge* of the vertex algebra. We can then identify each $\omega_{(p)} \in \mathrm{End}(V)$ as an action of $L_{p-1} \in \mathrm{Vir}$ on $V$. Moreover, one has

$$L_0|_{V_i} = i \cdot \mathrm{id}_V,\ \forall i \qquad \text{and} \qquad Y\left(L_{-1}A, z\right) = \partial_z Y(A,z).$$

The vertex operator of the conformal vector gives a representation of the Virasoro algebra on $V$, with the central element $K \in \mathrm{Vir}$ acting on $V$ as multiplication by the central charge $c$. The action of $L_0$ coincides with the *degree* operator on $V$; the action of $L_{-1}$ — called *translation* — is determined by $L_{-1}A = A_{(-2)}|0\rangle$, for $A \in V$.

As a consequence of the axioms, one has $A_{(i)}V_m \subseteq V_{m+\deg A - i-1}$, for homogeneous $A \in V$ (see e.g., [70]). We will then say that the degree of the operator $A_{(i)}$ is

$$\deg A_{(i)} := \deg A - i - 1, \qquad \text{for homogeneous } A \text{ in } V. \tag{1}$$

We will often abbreviate $(V, |0\rangle, \omega, Y(\cdot, z))$ with $V$.

1.3. **Modules of conformal vertex algebras.** Let $(V, |0\rangle, \omega, Y(\cdot, z))$ be a conformal vertex algebra. A *conformal $V$-module*, or simply a $V$-module, is a pair $(M, Y^M(\cdot, z))$, where:

(i) $M = \oplus_{i\geq 0} M_i$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$–vector space with $\dim M_i < \infty$;
(ii) $Y^M(\cdot, z)$ is a linear operation

$$\begin{aligned} Y^M(\cdot, z)\colon V &\to \mathrm{End}(M)[\![z, z^{-1}]\!] \\ A &\mapsto Y^M(A,z) := \sum_{i\in\mathbb{Z}} A^M_{(i)} z^{-i-1}. \end{aligned}$$

The pair $(M, Y^M(\cdot, z))$ is required to satisfy the following axioms:

(a) for all $A \in V$ and $v \in M$, one has $A^M_{(i)}v = 0$, for $i >> 0$;
(b) $Y^M(|0\rangle, z) = \mathrm{id}_M$;
(c) *(weak commutativity)* for all $A, B \in V$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $v \in M$ one has

$$(z-w)^N \left[Y^M(A,z), Y^M(B,w)\right] v = 0;$$

(d) *(weak associativity)* for all $A \in V$, $v \in M$, there exists $N \in \mathbb{Z}_{\geq 0}$ (depending only on $A$ and $v$) such that for all $B \in V$ one has

$$(w+z)^N \left(Y^M(Y(A,w)B, z) - Y^M(A, w+z)Y^M(B,z)\right) v = 0;$$

(e) the Fourier coefficients of $Y^M(\omega, z) = \sum_{i\in\mathbb{Z}} \omega^M_{(i)} z^{-i-1}$ satisfy the Virasoro relation

$$\left[\omega^M_{(p+1)}, \omega^M_{(q+1)}\right] = (p-q)\,\omega^M_{(p+q+1)} + \frac{c}{12}\,\delta_{p+q,0}\,(p^3-p)\,\mathrm{id}_M,$$

where $c \in \mathbb{C}$ is the central charge of $V$. We identify $\omega_{(p)} \in \mathrm{End}(M)$ with an action of $L_{p-1} \in \mathrm{Vir}$ on $M$. Moreover, $L_0$ acts semi-simply on $M$, and $Y^M\left(L_{-1}A, z\right) = \partial_z Y^M(a,z)$.

From these conditions it follows that if $A$ is a homogeneous element of $V$, then

$$A^M_{(i)} M_m \subseteq M_{m+\deg A - i - 1} \tag{2}$$

for all $m \geq 0$ and $i \in \mathbb{Z}$ (see e.g., [70]).

The Fourier coefficients of $Y^M(\omega, z)$ give an action of the Virasoro algebra on a $V$-module $M$, with the central element $K \in \mathrm{Vir}$ acting on $M$ as multiplication by the central charge $c$ of $V$.

After [53, Thm 3.5.4] or [35, §3.2.1], $V$ satisfies weak associativity. In particular, $V$ is a $V$-module.

We note that it has been shown that weak associativity and weak commutativity are equivalent to the *Jacobi identity* (see for instance [27], [38], [53], [54]).

1.3.1. A conformal $V$-module $M$ is *simple* if the only $V$-submodules of $M$ are itself and the 0 module. The *conformal dimension* $a$ of a simple V-module $M$ is defined by $L_0 v = (a + \deg v)v$, for homogeneous $v \in M$.

1.4. **Lie groups and Lie algebras.** Here we define a number of Lie groups and their associated Lie algebras related to automorphisms and derivations of $R[\![z]\!]$ and $R(\!(z)\!)$ for a $\mathbb{C}$-algebra $R$ and a formal variable $z$.

To begin with, we consider the group functor represented by a group ind-scheme denoted $\underline{\mathrm{Aut}}\,\mathcal{O}$ [35, §6.2.3]: this functor assigns to $R$ the Lie group of continuous automorphisms of $R[\![z]\!]$:

$$R \mapsto \left\{ z \mapsto \rho(z) = a_0 + a_1 z + a_2 z^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in R,\, a_1 \text{ a unit,} \\ a_0 \text{ nilpotent} \end{array} \right\}.$$

The group $\underline{\mathrm{Aut}}\,\mathcal{O}(R)$ parametrizes topological generators of $R[\![z]\!]$, that is, elements $t \in R[\![z]\!]$ such that $R[\![z]\!] \simeq R[\![t]\!]$.

Similarly, we consider the functor which assigns to $R$ the Lie group of continuous automorphisms of $R[\![z]\!]$ preserving the ideal $zR[\![z]\!]$:

$$R \mapsto \left\{ z \mapsto \rho(z) = a_1 z + a_2 z^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in R, \\ a_1 \text{ a unit} \end{array} \right\}.$$

This functor is represented by a group scheme denoted $\mathrm{Aut}\,\mathcal{O}$. The group $\mathrm{Aut}\,\mathcal{O}(R)$ parametrizes topological generators of $zR[\![z]\!]$: elements $t \in R[\![z]\!]$ such that $zR[\![z]\!] \simeq tR[\![t]\!]$.

Let $\mathrm{Aut}_+\mathcal{O}$ be the group scheme whose Lie group of $R$-points is

$$\mathrm{Aut}_+\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = z + a_2 z^2 + \cdots \mid a_i \in R \right\}.$$

Finally, consider the functor which assigns to $R$ the Lie group of continuous automorphisms of $R(\!(z)\!)$:

$$R \mapsto \left\{ z \mapsto \rho(z) = \sum_{i \geq i_0} a_i z^i \;\middle|\; \begin{array}{l} a_i \in R,\, a_1 \text{ a unit,} \\ a_i \text{ nilpotent for } i \leq 0 \end{array} \right\}.$$

This group functor is represented by a group ind-scheme $\mathrm{Aut}\,\mathcal{K}$ [35, §17.3.4].

One has the following inclusions:

$$\mathrm{Aut}_+\mathcal{O}(R) \subseteq \mathrm{Aut}\,\mathcal{O}(R) \subseteq \underline{\mathrm{Aut}}\,\mathcal{O}(R) \subseteq \mathrm{Aut}\,\mathcal{K}(R).$$

We will also consider the associated Lie algebras, obtained as the tangent space at the identity:

$$\begin{aligned} \mathrm{Der}_+\mathcal{O} &:= \mathrm{Lie}(\mathrm{Aut}_+\mathcal{O}), & \mathrm{Der}_0\,\mathcal{O} &:= \mathrm{Lie}(\mathrm{Aut}\,\mathcal{O}), \\ \mathrm{Der}\,\mathcal{O} &:= \mathrm{Lie}(\underline{\mathrm{Aut}}\,\mathcal{O}), & \mathrm{Der}\,\mathcal{K} &:= \mathrm{Lie}(\mathrm{Aut}\,\mathcal{K}), \end{aligned}$$

whose $R$-points are given by:

$$\begin{aligned} \mathrm{Der}_+\mathcal{O}(R) &= z^2 R[\![z]\!]\partial_z, & \mathrm{Der}_0\,\mathcal{O}(R) &= zR[\![z]\!]\partial_z, \\ \mathrm{Der}\,\mathcal{O}(R) &= R[\![z]\!]\partial_z, & \mathrm{Der}\,\mathcal{K}(R) &= R(\!(z)\!)\partial_z. \end{aligned}$$

The 2-cocycle defining the bracket of the Virasoro algebra $c(p,q) := \frac{1}{12}(p^3 - p)\delta_{p+q,0}$ vanishes for $p, q \geq -1$, hence the Lie algebras $\operatorname{Der}\mathcal{O}(R)$, $\operatorname{Der}_0\mathcal{O}(R)$, and $\operatorname{Der}_+\mathcal{O}(R)$ are Lie subalgebras of the Virasoro algebra $\operatorname{Vir}(R)$.

1.4.1. Observe that although one has the equalities

$$\operatorname{Aut}\mathcal{O}(\mathbb{C}) = \underline{\operatorname{Aut}}\,\mathcal{O}(\mathbb{C}) = \operatorname{Aut}\mathcal{K}(\mathbb{C}),$$

the $\mathbb{C}$-points of the associated Lie algebras have strict inclusions

$$z\mathbb{C}[\![z]\!]\partial_z \subset \mathbb{C}[\![z]\!]\partial_z \subset \mathbb{C}(\!(z)\!)\partial_z.$$

For instance, the tangent vector $\partial_z$ in $\operatorname{Der}\mathcal{O}(\mathbb{C})$ is the differential of an automorphism which is not detected over $\mathbb{C}$.

1.5. **The action of $\operatorname{Aut}_+\mathcal{O}$ on $V$-modules.** Let $V$ be a conformal vertex algebra, and let $M$ be a conformal $V$-module. The action of Vir induces actions of its Lie subalgebras $\operatorname{Der}\mathcal{O}$, $\operatorname{Der}_0\mathcal{O}$, and $\operatorname{Der}_+\mathcal{O}$ on $M$.

The Lie algebra $\operatorname{Der}_+\mathcal{O}$ is generated by $L_p$ with $p > 0$. After (1), each operator $L_p$ with $p > 0$ has degree $-p < 0$. Since the gradation on $M$ is bounded from below, that is, $M_i = 0$ for $i << 0$, the action of $\exp(L_p)$ is a finite sum, for $p > 0$, hence the action of $\operatorname{Der}_+\mathcal{O} = \operatorname{Lie}(\operatorname{Aut}_+\mathcal{O})$ can be exponentiated to a left action of $\operatorname{Aut}_+\mathcal{O}$ on $M$. Moreover, each $M_{\leq i} := \oplus_{m\leq i} M_m$ is a finite-dimensional $(\operatorname{Aut}_+\mathcal{O})$-submodule of $M$. The representation of $\operatorname{Aut}_+\mathcal{O}$ on $M$ is the inductive limit of the representations $M_{\leq i}$.

1.6. **The Lie algebra $\mathfrak{L}(V)$ associated to a vertex algebra $V$.** Given a vertex algebra $V$, define $\mathfrak{L}(V)$ as the quotient

$$\mathfrak{L}(V) := \big(V \otimes \mathbb{C}(\!(t)\!)\big)\big/\operatorname{Im}\partial \tag{3}$$

where $\partial := L_{-1} \otimes \operatorname{Id}_{\mathbb{C}(\!(t)\!)} + \operatorname{Id}_V \otimes \partial_t$. Denote by $A_{[i]}$ the projection in $\mathfrak{L}(V)$ of $A \otimes t^i \in V \otimes \mathbb{C}(\!(t)\!)$. The quotient $\mathfrak{L}(V)$ is a Lie algebra, with Lie bracket induced by

$$\left[A_{[i]}, B_{[j]}\right] := \sum_{k\geq 0} \binom{i}{k} \left(A_{(k)} \cdot B\right)_{[i+j-k]}.$$

The axiom on the vacuum vector $|0\rangle$ implies that $|0\rangle_{[-1]}$ is central in $\mathfrak{L}(V)$.

One has a Lie algebra homomorphism $\mathfrak{L}(V) \to \operatorname{End}(V)$: the element $A_{[i]}$ is mapped to the Fourier coefficient $A_{(i)}$ of the vertex operator $Y(A,z) = \sum_i A_{(i)} z^{-i-1}$. More generally, $\mathfrak{L}(V)$ is spanned by series of type $\sum_{i\geq i_0} f_i\, A_{[i]}$, for $A \in V$, $f_i \in \mathbb{C}$, and $i_0 \in \mathbb{Z}$; the series $\sum_{i\geq i_0} f_i\, A_{[i]}$ maps to

$$\operatorname{Res}_{z=0} Y(A,z) \sum_{i\geq i_0} f_i z^i dz$$

in $\operatorname{End}(V)$ [35, §4.1]. This defines an action of $\mathfrak{L}(V)$ on $V$.

Similarly to §1.4, one can extend $\mathfrak{L}(V)$ to a functor of Lie algebras assigning to a $\mathbb{C}$-algebra $R$ the Lie algebra $(V \otimes R(\!(t)\!))/\operatorname{Im}\partial$.

The Lie algebra $\mathfrak{L}(V)$ has a Lie subalgebra isomorphic to the Virasoro algebra: namely, the subalgebra generated by the elements

$$c \cdot |0\rangle_{[-1]} \simeq K \quad \text{and} \quad \omega_{[p]} \simeq L_{p-1}, \quad \text{for } p \in \mathbb{Z},$$

where $|0\rangle$ and $\omega$ are the vacuum and the conformal vector, respectively, and $c$ is the central charge of $V$. Via the above identification and the axiom on the vacuum vector, the central element $K \in \mathfrak{L}(V)$ acts on $V$ as multiplication by $c$. The action of $\mathfrak{L}(V)$ on $V$ extends then the action of Vir on $V$.

In particular, $\mathfrak{L}(V)$ has Lie subalgebras isomorphic to $\operatorname{Der}\mathcal{O}$, $\operatorname{Der}_0\mathcal{O}$, and $\operatorname{Der}_+\mathcal{O}$, as these are Lie subalgebras of the Virasoro algebra.

Given a conformal $V$-module $M$, there is a Lie algebra homomorphism $\mathfrak{L}(V) \to \operatorname{End}(M)$: the element $A_{[i]}$ is mapped to the Fourier coefficient $A^M_{(i)}$ of the vertex operator $Y^M(A,z)$. This defines an action of $\mathfrak{L}(V)$ on $M$ extending the action of Vir on $M$.

1.7. **Compatibility of actions of $\mathfrak{L}(V)$ and $\operatorname{Aut}_+\mathcal{O}$ on $V$-modules.** The left action of $\operatorname{Aut}_+\mathcal{O}$ on $M$ gives rise to a right action by $v \cdot \rho := \rho^{-1} \cdot v$, for $\rho \in \operatorname{Aut}_+\mathcal{O}$ and $v \in M$. The action of $\mathfrak{L}(V)$ on $M$ induces an anti-homomorphism of Lie algebras

$$\alpha_M \colon \mathfrak{L}(V) \to \operatorname{End}(M).$$

The *anti-homomorphism* $\alpha_M$ is compatible with the *right* action of $\operatorname{Aut}_+\mathcal{O}$ on $M$ in the following sense:

(i) the restriction of $\alpha_M$ to $\operatorname{Der}_+\mathcal{O}$ coincides with the differential of the right action of $\operatorname{Aut}_+\mathcal{O}$ on $M$ (equivalently, the restriction of $-\alpha_M$ to $\operatorname{Der}_+\mathcal{O}$ coincides with the differential of the left action of $\operatorname{Aut}_+\mathcal{O}$ on $M$);

(ii) for each $\rho \in \operatorname{Aut}_+\mathcal{O}$, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{L}(V) & \xrightarrow{\alpha_M} & \operatorname{End}(M) \\ {\scriptstyle \mathrm{Ad}_\rho}\big\downarrow & & \big\downarrow{\scriptstyle \rho_*} \\ \mathfrak{L}(V) & \xrightarrow{\alpha_M} & \operatorname{End}(M). \end{array}$$

Here, Ad is the adjoint representation of $\operatorname{Aut}_+\mathcal{O}$ on $\mathfrak{L}(V)$ induced from

$$\mathrm{Ad}_\rho\left(\sum_{i\in\mathbb{Z}} A_{[i]} z^{-i-1}\right) = \sum_{i\in\mathbb{Z}} \left(\rho_z^{-1}\cdot A\right)_{[i]} \rho(z)^{-i-1}$$

with $\rho_z(t) := \rho(z+t) - \rho(z)$ and $A \in V$. Finally, $\rho_*(-) := \rho^{-1}(-)\,\rho$.

Property (i) follows from the definition of the action of $\operatorname{Aut}_+\mathcal{O}$ on $M$ as integration of the action of $\operatorname{Der}_+\mathcal{O}$ on $M$. Property (ii) is due to Y.-Z. Huang [46, Prop. 7.4.1], [35, §17.3.13]:

$$\rho^{-1} Y^M(A,z)\rho = Y^M(\rho_z^{-1}\cdot A, \rho(z)), \quad \text{for } A \in V.$$

## 2. Virasoro uniformization for stable coordinatized curves

We present here the *Virasoro uniformization.* The statement is in [67], following prior work in [13], [22], [50]. For completeness, we prove the result which we use here, extending to families of stable curves with singularities the argument for families of smooth curves given in [35].

Let $\mathcal{C} \to S$ be a family of stable pointed curves over a smooth base $S$, together with $n$ sections $P_i \colon S \to \mathcal{C}$ and formal coordinates $t_i$ defined in a formal neighborhood of $P_i(S) \subset \mathcal{C}$. These data give rise to a moduli map $S \to \widehat{\mathcal{M}}_{g,n}$. Assume that each irreducible component of $C_s$ contains at least one marked point $P_i(s)$, for all $s \in S$. This ensures that $\mathcal{C} \setminus P_\bullet(S)$ is affine. Let $\Delta$ be the divisor of singular curves in $S$.

**Theorem 2.1** (Virasoro uniformization [13], [22], [50], [67])**.** *With notation as above, there exists an anti-homomorphism of Lie algebras*

$$\alpha\colon\ (\operatorname{Der}\mathcal{K}(\mathbb{C}))^{\oplus n}\ \widehat{\otimes}_{\mathbb{C}}\, H^0(S,\mathcal{O}_S) \to H^0\left(S, \mathcal{T}_S(-\log\Delta)\right)$$

*extending $\mathcal{O}_S$-linearly to a surjective anti-homomorphism of Lie algebroids (see §2.1), called the anchor map*

$$a\colon\ (\operatorname{Der}\mathcal{K}(\mathbb{C}))^{\oplus n}\ \widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S \twoheadrightarrow \mathcal{T}_S(-\log\Delta). \tag{4}$$

*The action of* $(\operatorname{Der}\mathcal{K})^n$ *on* $\widehat{\mathcal{M}}_{g,n}$ *induced from* $\alpha$ *is compatible with the action of* $(\operatorname{Aut}\mathcal{O})^n$ *on the fibers of* $\widehat{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}$*. The kernel of the anchor map* $a$ *is the subsheaf whose fiber at a point* $(C,P_\bullet,t_\bullet)$ *in* $S$ *is the Lie algebra* $\mathscr{T}_C(C\setminus P_\bullet)$ *of regular vector fields on* $C\setminus P_\bullet$*.*

Here and throughout, $\widehat{\otimes}$ is the completion of the usual tensor product with respect to the natural grading of $\operatorname{Der}\mathcal{K}(\mathbb{C})$. In particular, one has $(\operatorname{Der}\mathcal{K}(\mathbb{C}))^{\oplus n}\ \widehat{\otimes}_{\mathbb{C}}\,H^0(S,\mathcal{O}_S) = \left(\operatorname{Der}\mathcal{K}\left(H^0(S,\mathcal{O}_S)\right)\right)^{\oplus n}$ and $(\operatorname{Der}\mathcal{K}(\mathbb{C}))^{\oplus n}\ \widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S = \oplus_{i=1}^n\mathcal{O}_S(\!(t_i)\!)\partial_{t_i}$.

Our first application of Theorem 2.1 is to show that sheaves $\mathscr{M}_C$ of $V$-modules $M$ over a curve $C$, and sheaves $\mathscr{M}_{CJ}$ over $CJ$ (points of $CJ$ are pairs $(P,\tau)$, with $P\in C$, and $\tau$ a non-zero 1-jet at $P$, see §2.4) carry natural flat logarithmic connections (Propositions 3.1 and 3.2). When $M=V$, the sheaf $\mathscr{V}_C$ arises in the definition of the Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$. These results were known for smooth curves.

The structure of $\widehat{\mathcal{M}}_{g,n}$ is summarized in Figure 1 (see §2.2). The proof of Theorem 2.1 is contained in the remaining part of the section: the map $\alpha$ is described in §2.2.1, compatibility in §2.2.2. The space $\widehat{\mathcal{M}}_{g,n}$ is a principal $(\operatorname{Aut}\mathcal{O})^n$-bundle over $\overline{\mathcal{M}}_{g,n}$. We will also need the intermediate principal bundle $\overline{\mathcal{J}}^{1,\times}_{g,n}$, which is the moduli space of elements of type $(C,P_\bullet,\tau_\bullet=(\tau_1,\dots,\tau_n))$ such that $\tau_i$ a non-zero 1-jet of a formal coordinate at $P_i$ (see §2.3). In §2.5, we describe the restriction of the anchor map to a curve $C$ and discuss the uniformization of $\mathscr{A}ut_C$, the fiber of $\widehat{\mathcal{M}}_{g,1}\to\overline{\mathcal{M}}_g$.

$$\begin{tikzcd}
 & & \widehat{\mathcal{M}}_{g,n} \arrow[dl, "(\operatorname{Aut}_+\mathcal{O})^n"'] \arrow[dr, "(\operatorname{Aut}\mathcal{O})^n"] & & \\
\prod_{i=1}^n \mathscr{A}ut_{P_i,\tau_i} \arrow[urr, bend left] \arrow[dd] & \overline{\mathcal{J}}^{1,\times}_{g,n} \arrow[rr, "\mathbb{G}_m"] & & \overline{\mathcal{M}}_{g,n} & \prod_{i=1}^n \mathscr{A}ut_{P_i} \arrow[ull, bend right] \arrow[dd] \\
 & & & & \\
\operatorname{Spec}(\mathbb{C}) \arrow[uur, "(C,P_\bullet,\tau_\bullet)"] & & & & \operatorname{Spec}(\mathbb{C}) \arrow[uul, "(C,P_\bullet)"']
\end{tikzcd}$$

FIGURE 1. The structure of moduli spaces of coordinatized curves.

2.1. **Lie algebroids.** Following [21], we briefly review the definition of an Atiyah algebroid, referred to here. A logarithmic version is used in §6 and §7. Let $S$ be a scheme over $\mathbb{C}$. An *Atiyah algebroid* $\mathcal{A}$ over $S$ is a quasi-coherent $\mathcal{O}_S$-module together with a $\mathbb{C}$-linear bracket $[\cdot,\cdot]\colon\mathcal{A}\otimes_{\mathbb{C}}\mathcal{A}\to\mathcal{A}$ and an $\mathcal{O}_S$-module homomorphism $a\colon\mathcal{A}\to\mathscr{T}_S$ called the *anchor map*, for which: (i) $a$ is a Lie algebra homomorphism, and (ii) $[x_1,fx_2]=f[x_1,x_2]+(a(x_1)\cdot f)x_2$, for $x_1$ and $x_2$ in $\mathcal{A}$ and $f\in\mathcal{O}_S$. The tangent sheaf with the identity anchor map is the simplest example of an Atiyah algebroid.

2.2. **Description of $\widehat{\mathcal{M}}_{g,n}$.** The space $\widehat{\mathcal{M}}_{g,n}$ is a principal $(\mathrm{Aut}\,\mathcal{O})^n$-bundle over $\overline{\mathcal{M}}_{g,n}$. Indeed, consider the forgetful map $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. When $n=1$, the fiber over a $\mathbb{C}$-point $(C,P)$ of $\overline{\mathcal{M}}_{g,1}$ is the set of formal coordinates at $P$, i.e.,

$$\mathscr{A}ut_P := \left\{ t \in \widehat{\mathscr{O}}_P \,|\, t(P)=0,\, (dt)_P \neq 0 \right\}.$$

Here $\widehat{\mathscr{O}}_P$ is the completed local ring at the point $P$; after choosing a formal coordinate $t$ at $P$, one has $\widehat{\mathscr{O}}_P \simeq \mathbb{C}[\![t]\!]$. The set $\mathscr{A}ut_P$ admits a simply transitive right action of the group $\mathrm{Aut}\,\mathcal{O}(\mathbb{C})$ by change of coordinates:

$$\mathscr{A}ut_P \times \mathrm{Aut}\,\mathcal{O}(\mathbb{C}) \to \mathscr{A}ut_P, \qquad (t,\rho) \mapsto t\cdot\rho := \rho(t).$$

Elements of $\mathrm{Aut}\,\mathcal{O}(\mathbb{C})$ are power series $a_1 z + a_2 z^2 + \cdots$, such that $a_1 \neq 0$, and the group law is the composition of series. Thus, $\mathscr{A}ut_P$ is an $(\mathrm{Aut}\,\mathcal{O})$-torsor over a point. More generally, $\widehat{\mathcal{M}}_{g,n}$ is an $(\mathrm{Aut}\,\mathcal{O})^n$-torsor over $\overline{\mathcal{M}}_{g,n}$.

2.2.1. *Description of* $\alpha$. The map $\alpha$ can be constructed as the differential of the right action of the group ind-scheme $(\mathrm{Aut}\,\mathcal{K})^n$ on $\widehat{\mathcal{M}}_{g,n}$ preserving the divisor $\Delta$ (as in [35, §17.3.4]). As differential of a right action, the map $\alpha$ is an anti-homomorphism of Lie algebras, that is, $\alpha([l,m]) = -[\alpha(l),\alpha(m)]$.

The action is constructed as follows. The group ind-scheme $(\mathrm{Aut}\,\mathcal{K})^n$ acts naturally on the punctured formal discs of the $n$ marked points on stable $n$-pointed curves. Given a stable $n$-pointed curve $(C,P_\bullet)$ and an element $\rho \in (\mathrm{Aut}\,\mathcal{K})^n$, one glues $C \setminus P_\bullet$ and the formal discs of the $n$ marked points by twisting with the automorphism $\rho$ of the $n$ punctured formal discs to obtain a stable pointed curve. The resulting curve has the same topological type of the starting curve; in particular, the action preserves the divisor $\Delta$. This description of the action can be carried out in families, as in [35, §17.3.4]. As each irreducible component of a pointed curve $(C,P_\bullet)$ is assumed to have at least one marked point, the open curve $C\setminus P_\bullet$ is affine, as are the formal discs at the marked points. Since smooth affine varieties have no non-trivial infinitesimal deformations, all infinitesimal deformations are obtained by the differential of the action of $(\mathrm{Aut}\,\mathcal{K})^n$, hence the surjectivity of the anchor map $a$.

The map $\alpha$ also follows from a canonical map from $(\mathrm{Der}\,\mathcal{K})^n$ to the space of tangent directions preserving $\Delta$ at any point $(C,P_\bullet,t_\bullet)$ in $\widehat{\mathcal{M}}_{g,n}$. The Lie subalgebra $(\mathrm{Der}\,\mathcal{O})^n$ is canonically isomorphic to the space of tangent directions preserving the nodes of $C$ along the fiber of the forgetful map $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$, as in §2.5.1. Finally, the vector fields $t_i^p \partial_{t_i}$ with $p<0$ on the punctured disk around the point $P_i$ having a pole at $P_i$ correspond to infinitesimal changes of the complex structure on the curve $C$ preserving the topological type of $C$ (hence preserving $\Delta$).

2.2.2. The action of $(\mathrm{Der}\,\mathcal{K})^n$ on $\widehat{\mathcal{M}}_{g,n}$ via $\alpha$ is compatible with the right action of $(\mathrm{Aut}\,\mathcal{O})^n$ along the fibers of $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, that is:

(i) the restriction of $\alpha$ to the Lie subalgebra $(\mathrm{Der}_0\,\mathcal{O})^n = \mathrm{Lie}((\mathrm{Aut}\,\mathcal{O})^n)$ of $(\mathrm{Der}\,\mathcal{K})^n$ coincides with the differential of the right action of $(\mathrm{Aut}\,\mathcal{O})^n$ along the fibers of the principal $(\mathrm{Aut}\,\mathcal{O})^n$-bundle $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$;

(ii) for each $\rho \in (\mathrm{Aut}\,\mathcal{O})^n$, the following diagram commutes

$$\begin{array}{ccc}
(\mathrm{Der}\,\mathcal{K}(\mathbb{C}))^{\oplus n} \,\widehat{\otimes}_{\mathbb{C}}\, H^0(S,\mathcal{O}_S) & \xrightarrow{\ \alpha\ } & H^0\left(\widehat{\mathcal{M}}_{g,n}, \mathcal{T}_{\widehat{\mathcal{M}}_{g,n}}(-\log\Delta)\right) \\
{\scriptstyle \mathrm{Ad}_\rho}\big\downarrow & & \big\downarrow{\scriptstyle \rho_*} \\
(\mathrm{Der}\,\mathcal{K}(\mathbb{C}))^{\oplus n} \,\widehat{\otimes}_{\mathbb{C}}\, H^0(S,\mathcal{O}_S) & \xrightarrow{\ \alpha\ } & H^0\left(\widehat{\mathcal{M}}_{g,n}, \mathcal{T}_{\widehat{\mathcal{M}}_{g,n}}(-\log\Delta)\right).
\end{array}$$

That is, $\alpha(\mathrm{Ad}_\rho(\cdot)) = \rho_*(\alpha(\cdot))$. Here, Ad is the adjoint representation of $(\mathrm{Aut}\,\mathcal{O})^n$ on $(\mathrm{Der}\,\mathcal{K})^n$. Moreover, $\rho_*(\alpha(\cdot)) := \rho^{-1}\,\alpha(\cdot)\,\rho$.

2.3. **The moduli space $\overline{\mathcal{J}}^{1,\times}_{g,n}$.** A 1-jet at a smooth point $P$ on a stable curve is an equivalence class of functions for the relation: $\tau \sim \sigma$ if and only if $\tau - \sigma \in \mathfrak{m}_P^2$, for $\tau$ and $\sigma \in \widehat{\mathscr{O}}_P$, where $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathscr{O}}_P$. We say that $\tau$ is the 1-jet of $t \in \widehat{\mathscr{O}}_P$ if $\tau$ is the equivalence class represented by $t$. Let

$$\mathscr{A}ut_{P,\tau} := \{t \in \mathscr{A}ut_P \,|\, \tau \text{ is the 1-jet of t}\}.$$

This is an $(\mathrm{Aut}_+\mathcal{O})$-torsor over a point, with $\mathrm{Aut}_+\mathcal{O}$ acting on the right by change of coordinates. Recall that $\mathrm{Aut}_+\mathcal{O}$ is the subgroup of $\mathrm{Aut}\,\mathcal{O}$ of elements $\rho(z) = z + a_2 z^2 + \cdots$. One can show that $\mathrm{Aut}\,\mathcal{O} = \mathbb{G}_m \ltimes \mathrm{Aut}_+\mathcal{O}$.

Let $\overline{\mathcal{J}}^{1,\times}_{g,n}$ be the moduli space of objects of type $(C, P_\bullet, \tau_\bullet)$, where $(C, P_\bullet)$ is an $n$-pointed, genus $g$ stable curve and $\tau_\bullet = (\tau_1, \ldots, \tau_n)$ with each $\tau_i$ a non-zero 1-jet of a formal coordinate at $P_i$. The space $\overline{\mathcal{J}}^{1,\times}_{g,n}$ is a principal $(\mathbb{G}_m)^n$-bundle over $\overline{\mathcal{M}}_{g,n}$. Let $\Psi_i$ be the cotangent line bundle on $\overline{\mathcal{M}}_{g,n}$ corresponding to the $i$-th marked point. We identify $\overline{\mathcal{J}}^{1,\times}_{g,n}$ with the product of the principal $\mathbb{C}^\times$-bundles $\Psi_i \setminus \{\text{zero section}\}$ over $\overline{\mathcal{M}}_{g,n}$, for $i = 1, \ldots, n$.

There is a natural map $\widehat{\pi}\colon \widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}^{1,\times}_{g,n}$ obtained by mapping each local coordinate to its 1-jet. This realizes $\widehat{\mathcal{M}}_{g,n}$ as a principal $(\mathrm{Aut}_+\mathcal{O})^n$-bundle over $\overline{\mathcal{J}}^{1,\times}_{g,n}$. The action of $(\mathrm{Der}\,\mathcal{K})^n$ on $\widehat{\mathcal{M}}_{g,n}$ from (4) is compatible with the action of $(\mathrm{Aut}_+\mathcal{O})^n$ along the fibers of $\widehat{\pi}$, as in §2.2.2.

2.4. **Coordinatized curves.** Recall that the forgetful morphism $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ identifies the universal curve of $\overline{\mathcal{M}}_g$: we can then realize every stable curve $C$ as the fiber of that forgetful map over the Spec($\mathbb{C}$) point of $\overline{\mathcal{M}}_g$ corresponding to $C$. We can then define $\mathscr{A}ut_C$ as the fiber of the map $\widehat{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ over the natural map $C \to \overline{\mathcal{M}}_{g,1}$ (Figure 2). The bundle $\mathscr{A}ut_C$ is a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on $C$ whose fiber at a smooth $P \in C$ is $\mathscr{A}ut_P$. The description of the fiber over nodal points goes as follows: since $\overline{\mathcal{M}}_{g,1}$ is the universal curve over $\overline{\mathcal{M}}_g$ via stabilization, we replace the curve $C$ marked by the node $P$ with its stable reduction $(C', P')$ and then the fiber of $\mathscr{A}ut_C$ over $P$ is the space $\mathscr{A}ut_{P'}$. Equivalently, if $P$ is a node, we replace $C$ with its partial normalization $C^N$ at $P$ and consider the two points $P_+$ and $P_-$ lying above $P$. The fibers over $P_+$ and $P_-$ are $\mathscr{A}ut_{P_+}$ and $\mathscr{A}ut_{P_-}$, which are canonically isomorphic via the map sending $\rho$ to $\rho^{-1}$, and so they define the fiber of $\mathscr{A}ut_C$ at the node $P$. The bundle $\mathscr{A}ut_C$ is locally trivial in the Zariski topology.

$$\begin{tikzcd}
\widehat{\mathcal{M}}_{g,1} \arrow[d, "\mathrm{Aut}\,\mathcal{O}"'] \arrow[dr, "\mathrm{Aut}_+\mathcal{O}"] & & & \mathscr{A}ut_C \arrow[lll] \arrow[d] \arrow[dr] & & \\
& \overline{\mathcal{J}}^{1,\times}_{g,1} \arrow[dl, "\mathbb{G}_m"] & & \phantom{x} \arrow[d] & CJ \arrow[lll] \arrow[dl] & \\
\overline{\mathcal{M}}_{g,1} \arrow[drr, bend right] & & & C \arrow[lll] \arrow[drr, bend right] & & \\
& & \overline{\mathcal{M}}_g & & & \mathrm{Spec}(\mathbb{C}) \arrow[lll]
\end{tikzcd}$$

FIGURE 2. The definition of $\mathscr{A}ut_C$ and $CJ$.

Similarly, let $CJ$ be the fiber of the forgetful map $\overline{\mathcal{J}}_{g,1}^{1,\times} \to \overline{\mathcal{M}}_{g,1}$ over the point $C$ in $\overline{\mathcal{M}}_{g,1}$. The space $CJ$ is a principal $\mathbb{G}_m$-bundle on $C$, whose points are pairs $(P, \tau)$, with $P$ a stable point in $C$, and $\tau$ a non-zero 1-jet of a formal coordinate at $P$. Mapping a formal coordinate to its 1-jet realizes $\mathscr{A}ut_C$ as a principal $(\mathrm{Aut}_+\mathcal{O})$-bundle on $CJ$, whose fibers at $(P, \tau)$ is $\mathscr{A}ut_{P,\tau}$, as pictured in Figure 3.

$$\begin{tikzcd}
 & & \mathscr{A}ut_C \arrow[ddl, "\mathrm{Aut}_+\mathcal{O}"'] \arrow[ddr, "\mathrm{Aut}\,\mathcal{O}"] & & \\
\mathscr{A}ut_{P,\tau} \arrow[urr, bend left, hook] \arrow[dd] & & & & \mathscr{A}ut_P \arrow[ull, bend right, hook'] \arrow[dd] \\
 & CJ \arrow[rr, "\mathbb{G}_m"] & & C & \\
(P,\tau) \arrow[ur, hook] \arrow[rrrr, mapsto] & & & & P \arrow[ul, hook']
\end{tikzcd}$$

FIGURE 3. The structure of $\mathscr{A}ut_C$.

2.5. **Uniformization of $\mathscr{A}ut_C$.** As described in [35, §17.1], the Lie algebra $\mathrm{Der}\,\mathcal{O}$ has a simply transitive action on the space $\mathscr{A}ut_C$ over a smooth curve $C$. Here we discuss the action, and generalize it to the case of stable curves. Given a stable curve $C$, let $D$ be the divisor in $\mathscr{A}ut_C$ lying over the singular locus of $C$. Let $\mathscr{T}_{\mathscr{A}ut_C}(-\log D)$ be the sheaf of $\mathbb{C}$-linear derivations of $\mathscr{A}ut_C$ which preserve the ideal defining $D$. There is an anti-homomorphism of Lie algebras

$$\alpha_C \colon \mathrm{Der}\,\mathcal{O}(\mathbb{C}) \,\widehat{\otimes}_{\mathbb{C}}\, H^0(\mathscr{A}ut_C, \mathscr{O}_{\mathscr{A}ut_C}) \to H^0\left(\mathscr{A}ut_C, \mathscr{T}_{\mathscr{A}ut_C}(-\log D)\right) \tag{5}$$

described below in §2.5.1, that extends $\mathscr{O}_{\mathscr{A}ut_C}$-linearly to an anti-isomorphism of Lie algebroids

$$a_C \colon \mathrm{Der}\,\mathcal{O}(\mathbb{C}) \,\widehat{\otimes}_{\mathbb{C}}\, \mathscr{O}_{\mathscr{A}ut_C} \xrightarrow{\sim} \mathscr{T}_{\mathscr{A}ut_C}(-\log D). \tag{6}$$

The restrictions of the maps $\alpha$ and $a$ from Theorem 2.1 to the fiber of the forgetful map $\widehat{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ over a point $C$ in $\overline{\mathcal{M}}_g$ coincide with the maps $\alpha_C$ and $a_C$, respectively.

2.5.1. *Description of $\alpha_C$.* A heuristic description of the map $\alpha_C$ as a bijection of vector spaces can be given by showing how to each element of $\mathrm{Der}\,\mathcal{O}$ corresponds a vector field on $\mathscr{A}ut_C$ preserving the divisor $D$. Since a vector field is the choice of a tangent vector at each point, it is enough to assign to each element of $\mathrm{Der}\,\mathcal{O}$ a tangent vector preserving $D$ at each point in $\mathscr{A}ut_C$.

Consider the following diagrams:

$$\begin{tikzcd}
\mathscr{A}ut_C \arrow[r, "f"] \arrow[d, "p"] & \mathrm{Aut}_+\mathcal{O} \arrow[d] \\
CJ \arrow[r] & \mathrm{Spec}(\mathbb{C}),
\end{tikzcd}
\qquad
\begin{tikzcd}
CJ \arrow[r, "h"] \arrow[d, "q"] & \mathbb{G}_m \arrow[d] \\
C \arrow[r] & \mathrm{Spec}(\mathbb{C}).
\end{tikzcd}$$

Since $\mathscr{A}ut_C$ (resp. $CJ$) is a principal $(\mathrm{Aut}_+\mathcal{O})$-bundle (resp. $\mathbb{G}_m$-bundle), locally the two diagrams are cartesian. This implies that we can decompose the tangent sheaf of $\mathscr{A}ut_C$ and $CJ$ as the following direct sums

$$\mathscr{T}_{\mathscr{A}ut_C}(-\log D) = p^*\mathscr{T}_{CJ}(-\log D) \oplus f^*\mathscr{T}_{\mathrm{Aut}_+\mathcal{O}},$$
$$\mathscr{T}_{CJ}(-\log D) = q^*\mathscr{T}_{C}(-\log D) \oplus h^*\mathscr{T}_{\mathbb{G}_m}.$$

Observe that the elements of $\mathscr{T}_C$ are derivations of $\mathscr{O}_C$ which preserve the ideal defining the singular points, so we can rewrite the above equality as

$$\mathscr{T}_{CJ}(-\log D) = q^*\mathscr{T}_C \oplus h^*\mathscr{T}_{\mathbb{G}_m}.$$

Combining the two decompositions, we obtain

$$\mathscr{T}_{\mathscr{A}ut_C}(-\log D) = p^*q^*\mathscr{T}_C \oplus p^*h^*\mathscr{T}_{\mathbb{G}_m} \oplus f^*\mathscr{T}_{\mathrm{Aut}_+\mathcal{O}}.$$

From this characterization, it follows that the tangent space of $\mathscr{T}_{\mathscr{A}ut_C}$ at a point $(P,t)$, where $P$ is a smooth point of $C$, can be described as the sum

$$T_{C,P} \oplus T_{\mathbb{G}_m} \oplus T_{\mathrm{Aut}_+\mathcal{O}}$$

which is isomorphic to

$$\mathbb{C}\partial_t \oplus \mathbb{C}t\partial_t \oplus \mathrm{Der}_+\mathcal{O}(\mathbb{C}) = \mathrm{Der}\,\mathcal{O}(\mathbb{C}). \tag{7}$$

The space $\mathbb{C}t\partial_t$ corresponds to infinitesimal changes of the 1-jet $\tau$ of $t$ fixing the point $P$. In the same spirit, the space $\mathbb{C}\partial_t$ is identified with the tangent direction at $(P,t)$ corresponding to infinitesimal changes of the point $P$ on the curve.

Note that when $P$ is non smooth, we can replace $(C,P)$ with its stable reduction $(C',P')$ where $P'$ lies in a rational component of $C'$. Also in this case we obtain that the space is isomorphic to (7). In this case, however, infinitesimal changes of the point $P'$ on the rational component are identified by the automorphisms of the rational component, hence the tangent space $\mathbb{C}\partial_t$ is zero at $P'$. It follows that all tangent directions described above preserve the singular locus $D$ in $\mathscr{A}ut_C$.

From this identification of $\mathrm{Der}\,\mathcal{O}$ with the space of tangent directions preserving $D$ at any point in $\mathscr{A}ut_C$, an element of $\mathrm{Der}\,\mathcal{O}$ gives rise to a tangent vector at each point of $\mathscr{A}ut_C$, hence a vector field on $\mathscr{A}ut_C$ preserving $D$.

More precisely, the map $\alpha_C$ is given by taking the differential of the right action of the exponential of $\mathrm{Der}\,\mathcal{O}$. Recall that the exponential of $\mathrm{Der}\,\mathcal{O}$ is the group ind-scheme $\underline{\mathrm{Aut}}\,\mathcal{O}$ (see §1.4). Consider the principal $(\underline{\mathrm{Aut}}\,\mathcal{O})$-bundle

$$\underline{\mathscr{A}ut}_C := \underline{\mathrm{Aut}}\,\mathcal{O} \underset{\mathrm{Aut}\,\mathcal{O}}{\times} \mathscr{A}ut_C$$

on $C$. For a $\mathbb{C}$-algebra $R$, an $R$-point of $\underline{\mathscr{A}ut}_C$ is a pair $(P,t)$, where $P$ is an $R$-point of $C$, and $t$ is an element of $R\widehat{\otimes}\widehat{\mathscr{O}}_P$ such that there is a continuous isomorphism of algebras $R\widehat{\otimes}\widehat{\mathscr{O}}_P \simeq R[\![t]\!]$. The group ind-scheme $\underline{\mathrm{Aut}}\,\mathcal{O}$ has a right action on $\underline{\mathscr{A}ut}_C$, and the differential of this action gives the map $\alpha_C$. The argument is presented in [35, §17.1.3] for smooth curves and generalizes to nodal curves.

## 3. $V$-module sheaves on curves and their flat logarithmic connections

Here given a $V$-module $M$, we define sheaves $\mathscr{M}_C$ over a curve $C$, and $\mathscr{M}_{CJ}$ over $CJ$ (points of $CJ$ are pairs $(P,\tau)$, with $P\in C$, and $\tau$ a non-zero 1-jet at $P$, see §2.4). The most important for our applications is the sheaf $\mathscr{V}_C$, where $V$ is considered as a module over itself. We show that $\mathscr{M}_{CJ}$ and $\mathscr{V}_C$ have flat logarithmic connections (Propositions 3.1 and 3.2). These results have been proved for smooth curves, and here we extend the results to stable curves with singularities using the Virasoro uniformization (Theorem 2.1).

Proposition 3.2 is used to define vector spaces of coinvariants and their sheaves (see §4 and §5). While one does not obtain a projectively flat connection on $V$-module sheaves in families, one does obtain a twisted logarithmic $\mathcal{D}$-module structure on certain quotients, giving rise to projectively flat connections on the sheaves of coinvariants (Theorems 6.1 and 7.1).

The structure of a $V$-module bundle is summarized in Figure 4.

$$\begin{tikzcd}
 & M_{CJ} \arrow[r] \arrow[dd] & M_C \arrow[dd] & \\
M_{P,\tau} \arrow[ur, bend left] \arrow[dd] & & & M_P \arrow[ul, bend right] \arrow[dd] \\
 & CJ \arrow[r, "\mathbb{G}_m"] & C & \\
(P,\tau) \arrow[ur, bend left] \arrow[rrr, mapsto] & & & P \arrow[ul, bend right]
\end{tikzcd}$$

FIGURE 4. The structure of a $V$-module bundle.

3.1. **The sheaf $\mathscr{M}_{CJ}$.** In §3.1.1 we describe the sheaves $\mathscr{M}_{CJ}$, and in §3.1.2 we define the connection $\nabla$ which will be used to define the sheaves of coinvariants in §4.

3.1.1. *Description.* Let $V$ be a conformal vertex algebra, and $M = \oplus_{i\geq 0} M_i$ a conformal $V$-module. Let $(C, P)$ be a stable pointed curve. Consider the trivial vector bundle

$$M_{\mathscr{A}ut_C} := \mathscr{A}ut_C \times \varinjlim_i M_{\leq i} = \varinjlim_i \mathscr{A}ut_C \times M_{\leq i}$$

on $\mathscr{A}ut_C$, where we still denote by $M_{\leq i}$ the finite-dimensional affine complex space associated with the $\mathbb{C}$-vector space $M_{\leq i}$. A $\mathbb{C}$-point of $M_{\mathscr{A}ut_C}$ is a triple $(P, t, m)$, where $P$ is a $\mathbb{C}$-point of the curve $C$, $t$ is a local coordinate at $P$, and $m$ is an element of the module $M$. Observe that for an infinite-dimensional vector space $M$, the vector bundle $M_{\mathscr{A}ut_C}$ is not an affine scheme over $\mathscr{A}ut_C$, but an ind-scheme. By abuse of notation, we will simply write

$$M_{\mathscr{A}ut_C} := \mathscr{A}ut_C \times M.$$

The sheaf of sections $\mathscr{M}_{\mathscr{A}ut_C}$ of the natural map $M_{\mathscr{A}ut_C} \to \mathscr{A}ut_C$ is locally free of infinite rank.

As discussed in §2.4, $\mathscr{A}ut_C$ is a principal $(\mathrm{Aut}_+\mathcal{O})$-bundle over $CJ$, hence has a right action of $\mathrm{Aut}_+\mathcal{O}$. The left action of $\mathrm{Aut}_+\mathcal{O}$ on the modules $M_{\leq i}$, as in §1.5, induces a right action of $\mathrm{Aut}_+\mathcal{O}$ on $M_{\leq i}$, as in §1.7. It follows that $\mathscr{A}ut_C \times M_{\leq i}$ has an equivariant right action of $\mathrm{Aut}_+\mathcal{O}$. This action is compatible with the inductive limit, so it induces an equivariant right action of $\mathrm{Aut}_+\mathcal{O}$ on $M_{\mathscr{A}ut_C}$. On the $\mathbb{C}$-points of $M_{\mathscr{A}ut_C}$, the action of $\mathrm{Aut}_+\mathcal{O}$ is:

$$\begin{array}{ccc} M_{\mathscr{A}ut_C}(\mathbb{C}) \times \mathrm{Aut}_+\mathcal{O}(\mathbb{C}) & \to & M_{\mathscr{A}ut_C}(\mathbb{C}) \\ (P, t, m) \times \rho & \mapsto & \left(P, t\cdot\rho, \rho^{-1}\cdot m\right). \end{array} \tag{8}$$

The quotient of $M_{\mathscr{A}ut_C}$ by this action descends to a vector bundle on $CJ = \mathscr{A}ut_C/\mathrm{Aut}_+\mathcal{O}$:

$$M_{CJ} := \mathscr{A}ut_C \underset{\mathrm{Aut}_+\mathcal{O}}{\times} M.$$

Recall that $p\colon \mathscr{A}ut_C \to CJ$ is the natural projection. The sheaf of sections

$$\mathscr{M}_{CJ} := \left(M \otimes p_* \mathscr{O}_{\mathscr{A}ut_C}\right)^{\mathrm{Aut}_+\mathcal{O}}$$

of $M_{CJ}$ is then a locally free $\mathscr{O}_{CJ}$-module of infinite type.

The fibers of $M_{CJ}$ on $CJ$ are described as follows. The fiber of $M_{\mathscr{A}ut_C}$ over a point $(P,\tau)\in CJ(\mathbb{C})$ is the trivial bundle $\mathscr{A}ut_{P,\tau}\times M$ on $\mathscr{A}ut_{P,\tau}$. The group $\mathrm{Aut}_+\mathcal{O}$ acts equivariantly on $\mathscr{A}ut_{P,\tau}\times M$, as in (8). The fiber of $M_{CJ}$ over $(P,\tau)\in CJ(\mathbb{C})$ is the space

$$M_{P,\tau} := \mathscr{A}ut_{P,\tau} \underset{\mathrm{Aut}_+\mathcal{O}}{\times} M \tag{9}$$

defined as the quotient of $\mathscr{A}ut_{P,\tau}\times M$ modulo the relations

$$(t\cdot\rho, m) = (t, \rho^{-1}\cdot m), \quad \text{for } \rho\in \mathrm{Aut}_+\mathcal{O}(\mathbb{C}) \text{ and } (t,m)\in\mathscr{A}ut_{P,\tau}\times M.$$

One can identify $M_{P,\tau}$ with $\{(\tau,m)\,|\, m\in M\}\simeq M$. Thus we regard $M_{P,\tau}$ as a realization of the module $M$ assigned at the pair $(P,\tau)$. It is independent of the curve $C$.

3.1.2. *The flat logarithmic connection on $\mathscr{M}_{CJ}$.* The strategy to produce a flat logarithmic connection on $\mathscr{M}_{CJ}$ is to first construct a flat logarithmic connection on the trivial bundle $\mathscr{A}ut_C\times M$ on $\mathscr{A}ut_C$ induced by the action of the Lie algebra $\mathrm{Der}\,\mathcal{O}$. Since the action of $\mathrm{Der}\,\mathcal{O}$ is shown to be $(\mathrm{Aut}_+\mathcal{O})$-equivariant in the sense of §2.2.2, the connection then descends to $\mathscr{M}_{CJ}$. For a smooth curve, this has been treated in [35, §17.1].

As discussed in §1.7, the action of $\mathfrak{L}(V)$ on $M$ gives rise to an anti-homomorphism of Lie algebras

$$\alpha_M\colon \mathrm{Der}\,\mathcal{O}(\mathbb{C})\to\mathrm{End}(M).$$

The map $\alpha_M$ extends to an anti-homomorphism of sheaves of Lie algebras

$$\beta_C\colon \mathrm{Der}\,\mathcal{O}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathscr{O}_{\mathscr{A}ut_C}\to\mathrm{End}(M)\otimes_{\mathbb{C}}\mathscr{O}_{\mathscr{A}ut_C} \tag{10}$$

defined by

$$l\otimes f\mapsto \alpha_M(l)\otimes f + \mathrm{id}_M\otimes(\alpha_C(l)\cdot f), \quad \text{for } l\in\mathrm{Der}\,\mathcal{O}(\mathbb{C}) \text{ and } f\in\mathscr{O}_{\mathscr{A}ut_C}.$$

From (5), $\alpha_C$ maps an element of $\mathrm{Der}\,\mathcal{O}$ to a vector field on $\mathscr{A}ut_C$, hence the action on regular functions above is by derivations. Composing with (6), the map (10) gives rise to a homomorphism of sheaves of Lie algebras

$$\mathscr{T}_{\mathscr{A}ut_C}(-\log D)\to\mathrm{End}(M)\otimes_{\mathbb{C}}\mathscr{O}_{\mathscr{A}ut_C} \tag{11}$$

where the target is the sheaf of endomorphisms of $\mathscr{A}ut_C\times M$, and $D$ is the divisor in $\mathscr{A}ut_C$ lying over the singular locus in $C$.

The properties of the maps $\alpha_C$ and $\alpha_M$ described respectively in §2.2.2 and §1.7 imply that the map (11) is $(\mathrm{Aut}_+\mathcal{O})$-equivariant. This is one of the main ingredients to deduce the following statement:

**Proposition 3.1.** *The sheaf $\mathscr{M}_{CJ}$ is naturally equipped with a flat logarithmic connection, i.e., a homomorphism of Lie algebras*

$$\mathscr{T}_{CJ}(-\log D)\to \mathit{End}\,(\mathscr{M}_{CJ})\,. \tag{12}$$

*Proof.* Consider the exact sequence involving the tangent sheaves of $\mathscr{A}ut_C$ and $CJ$:

$$0\to\mathscr{T}^{\mathrm{vert}}_{\mathscr{A}ut_C}\to\mathscr{T}_{\mathscr{A}ut_C}(-\log D)\to p^*\mathscr{T}_{CJ}(-\log D)\to 0. \tag{13}$$

From the isomorphism $\mathscr{T}_{\mathscr{A}ut_C}(-\log D)\simeq\mathrm{Der}\,\mathcal{O}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathscr{O}_{\mathscr{A}ut_C}$ and the fact that the elements $t^i\partial_t\otimes f$ are killed in $p^*\mathscr{T}_{CJ}(-\log D)$ if $i\geq 2$, we deduce that

$$\mathscr{T}^{\mathrm{vert}}_{\mathscr{A}ut_C}\simeq\mathrm{Der}_+\mathcal{O}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathscr{O}_{\mathscr{A}ut_C}.$$

In particular this implies that pushing forward the exact sequence (13) and taking $(\mathrm{Aut}_+\mathcal{O})$-invariants, we can describe $\mathscr{T}_{CJ}(-\log D)$ as the quotient

$$\left(\mathrm{Der}\,\mathcal{O}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\, p_*\mathscr{O}_{\mathscr{A}ut_C}\right)^{\mathrm{Aut}_+\mathcal{O}} / \left(\mathrm{Der}_+\mathcal{O}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\, p_*\mathscr{O}_{\mathscr{A}ut_C}\right)^{\mathrm{Aut}_+\mathcal{O}}.$$

In order to induce a connection on $\mathscr{M}_{CJ}$ from the one on $\mathscr{M}$, it is sufficient to prove that $(\mathrm{Der}_+\mathcal{O}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathscr{O}_{\mathscr{A}ut_C})^{\mathrm{Aut}_+\mathcal{O}}$ acts trivially on $\mathscr{M}_{CJ}$. By definition, the action of $\mathscr{T}^{\mathrm{vert}}_{\mathscr{A}ut_C}$ on $M \otimes \mathscr{O}_{\mathscr{A}ut_C}$ is obtained by differentiating the natural action of $\mathrm{Aut}_+\mathcal{O}$. Since $\mathscr{M}_{CJ}$ is given by the elements of $\mathscr{M}_{\mathscr{A}ut_C}$ on which $\mathrm{Aut}_+\mathcal{O}$ acts as the identity, then the associated Lie algebra action will be trivial, concluding the argument. □

For the reader familiar with the localization of modules of Harish-Chandra pairs ([21]; see also [19, §3]), the construction in this section is the result of the localization with respect to the Harish-Chandra pair $(\mathrm{Der}\,\mathcal{O}, \mathrm{Aut}_+\mathcal{O})$: the $(\mathrm{Der}\,\mathcal{O}, \mathrm{Aut}_+\mathcal{O})$-module $M$ is transformed to the logarithmic $\mathcal{D}$-module $\mathscr{M}_{CJ}$ on $CJ$.

3.2. **The sheaf $\mathscr{M}_C$.** In §3.2.2 we describe the sheaf $\mathscr{M}_C$, and in §3.2.4, using the flat logarithmic connection on $\mathscr{V}_{CJ}$, we show there is a flat logarithmic connection on $\mathscr{V}_C$.

3.2.1. *The case $M = V$.* When $M = V$, the action of $L_0 = -z\partial_z$ defines the integral gradation of $V$: $v \in V_i$ if and only if $L_0(v) = iv$. The integral gradation gives rise to a $\mathbb{C}^\times$-action on $V$, as in (15). It follows that the action of $\mathrm{Der}_0\,\mathcal{O} = \mathrm{Lie}(\mathrm{Aut}\,\mathcal{O})$ on $V$ from §1.6 can be integrated to an action of $\mathrm{Aut}\,\mathcal{O} = \mathbb{G}_m \ltimes \mathrm{Aut}_+\mathcal{O}$ on $V$. Replacing $\mathrm{Aut}_+\mathcal{O}$ with $\mathrm{Aut}\,\mathcal{O}$ in §3.1 and §3.1.2, one produces the *vertex algebra bundle*

$$V_C := \mathscr{A}ut_C \underset{\mathrm{Aut}\,\mathcal{O}}{\times} V$$

on $C$, whose sheaf of sections

$$\mathscr{V}_C := \left(V \otimes q_*p_*\mathscr{O}_{\mathscr{A}ut_C}\right)^{\mathrm{Aut}\,\mathcal{O}}$$

is a quasi-coherent locally free $\mathscr{O}_C$-module. Here $p\colon \mathscr{A}ut_C \to CJ$ and $q\colon CJ \to C$ are the natural projections. Being $(\mathrm{Aut}\,\mathcal{O})$-equivariant, the connection from (11) gives rise to a flat logarithmic connection

$$\mathscr{T}_{CJ}(-\log D) \otimes \mathscr{V}_C \to \mathscr{V}_C. \tag{14}$$

3.2.2. *$\mathbb{C}^\times$-equivariance.* In general, the action of $L_0$ cannot be integrated to an action of $\mathbb{C}^\times$, unless the action of $L_0$ has only integral eigenvalues.

We obtain an action of $\mathbb{C}^\times$ in a different way. Our assumption is that a $V$-module $M$ is $\mathbb{Z}$-graded, with gradation bounded from below. The $\mathbb{Z}$-gradation induces an action of $\mathbb{C}^\times$ on $M$:

$$z \cdot v := z^{-\deg v} v, \qquad \text{for } z \in \mathbb{C}^\times \text{ and homogeneous } v \in M. \tag{15}$$

This allows one to define a $\mathbb{C}^\times$-equivariant action on $M_{CJ} \to CJ$:

$$(\tau, v) \cdot z := (\tau \cdot z, z^{-1} \cdot v), \qquad \text{for } z \in \mathbb{C}^\times,\ \tau \neq 0 \text{ a 1-jet, and } v \in M.$$

As $q\colon CJ \to C$ is a principal $\mathbb{G}_m$-bundle on the curve $C$, the bundle $M_{CJ}$ descends to a bundle $M_C$ on $C$:

$$M_C := \left(q_*M_{CJ}\right)^{\mathbb{C}^\times}.$$

Namely, $M_C$ is the quotient of $M_{CJ}$ by the relations

$$(\tau \cdot z, v) = (\tau, z^{-\deg v} v), \qquad \text{for } z \in \mathbb{C}^\times,\ \tau \neq 0 \text{ a 1-jet, and homogeneous } v \in M.$$

This is the largest $\mathbb{C}^\times$-invariant quotient of $M_{CJ}$. We denote by $\mathscr{M}_C$ the sheaf of sections of $M_C$:

$$\mathscr{M}_C := (q_* \mathscr{M}_{CJ})^{\mathbb{C}^\times} ,$$

and by $M_P$ the fiber of $\mathscr{M}_C$ over a point $P \in C$.

Note that when $L_0(v) = (\deg v)v$ for homogeneous $v \in M$ (e.g., $M = V$), the $L_0$-action on $M$ integrates to the $\mathbb{C}^\times$-action in (15) on $M$. In general, the eigenvalues of $L_0$ are complex numbers, and thus $L_0(v) \neq (\deg v)v \in \mathbb{Z}v$.

3.2.3. *Description of the action of $\mathbb{G}_m$ on the sheaf of sections.* We know that locally on $CJ$ the sheaf $\mathscr{M}_{CJ}$ is locally free, hence isomorphic to $M \otimes \mathscr{O}_{CJ}$. Moreover, we can further assume that $CJ = \operatorname{Spec}(R[w, w^{-1}])$ where $R$ is the coordinate ring of $C$. The action of an element $z \in \mathbb{C}^\times$ on $\mathscr{M}_{CJ}$ is then given by the morphism

$$M \otimes R[w, w^{-1}] \to M \otimes R[w, w^{-1}], \quad v \otimes w^n \mapsto z^{-\deg v + n} v \otimes w^n.$$

This description implies that the invariants under this action are linear combinations of elements of the type $v \otimes w^{\deg v}$ for $v \in M$. We will use this description in §7.

3.2.4. *The connection on $\mathscr{V}_C$.* With an argument analogous to the proof of Proposition 3.1, one can show that $\mathscr{V}_C$ has a flat logarithmic connection:

**Proposition 3.2.** *The sheaf $\mathscr{V}_C$ is naturally equipped with a flat logarithmic connection, i.e., there is a morphism*

$$\nabla : \mathscr{V}_C \to \mathscr{V}_C \otimes \omega_C \tag{16}$$

*arising from an action of the Lie algebra $\mathscr{T}_C$ on $\mathscr{V}_C$.*

*Proof.* The action of the tangent sheaf $\mathscr{T}_{CJ}(-\log D)$ on $\mathscr{M}_{CJ}$ is $\mathbb{C}^\times$-equivariant, so we obtain that $\mathscr{T}_{CJ}(-\log(D))^{\mathbb{C}^\times}$ acts on $\mathscr{M}_C$. We are interested in $M = V$, in which case the action of $L_0$ on $V$ can be integrated to an action of $\mathbb{C}^\times$, hence, using an argument similar to the one given for the proof of Proposition 3.1, it follows that $\mathscr{T}_C(-\log D)$, which coincides with $\mathscr{T}_C$, acts on $\mathscr{V}_C$. For what follows it is more convenient to describe this action on $\mathscr{V}_C$ in terms of differentials: we can rewrite the natural action of $\mathscr{T}_C$ on $\mathscr{V}_C$ as a morphism

$$\Omega_C^\vee \otimes \mathscr{V}_C \to \mathscr{V}_C$$

that via the canonical map $\Omega_C \to \omega_C$, induces a map $\omega_C^\vee \otimes \mathscr{V}_C \to \mathscr{V}_C$, equivalent to $\nabla$ in (16). $\square$

## 4. Spaces of coinvariants

For a stable $n$-pointed curve $(C, P_\bullet)$ and $V$-modules $M^\bullet = (M^1, \ldots, M^n)$, we describe vector spaces of coinvariants $\mathbb{V}^J(V; M^\bullet)_{(C,P_\bullet,\tau_\bullet)}$ at $(C, P_\bullet, \tau_\bullet)$ (dependent on a choice of non-zero 1-jets $\tau_\bullet$ at the marked points, see (21)), and $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)}$ at $(C, P_\bullet)$ (independent of jets, see (22)) given by the action of the Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$ associated to $(C, P_\bullet)$ and the conformal vertex algebra $V$ (see (17)). Recall that for a representation $\mathbb{M}$ of $\mathscr{L}_{C\setminus P_\bullet}(V)$, the *space of coinvariants* of $\mathbb{M}$ is the quotient $\mathbb{M}/\mathscr{L}_{C\setminus P_\bullet}(V) \cdot \mathbb{M}$; this is the largest quotient of $\mathbb{M}$ on which $\mathscr{L}_{C\setminus P_\bullet}(V)$ acts trivially.

As explained in §4.2, for each marked point $P_i$, the Lie algebra $\mathscr{L}_{P_i}(V)$ (see (18)) gives a coordinate-free realization of $\mathfrak{L}(V)$ (see (3)) based at $P_i$ by [35, Cor. 19.4.14]. Following Lemma 4.1, the Lie algebra $\mathscr{L}_{P_i}(V)$ acts on the coordinate-free realization of the $V$-module $M^i$ based at $P_i$. In §4.3 we will see that this induces an action of $\mathscr{L}_{C\setminus P_\bullet}(V)$ on the coordinate-free realization of $\otimes_i M^i$ based at the marked points $P_\bullet$.

Finally, in §4.4, we gather a few statements concerning the Virasoro vertex algebra $\mathrm{Vir}_c$ and the action of $\mathscr{T}_C(C \setminus P_\bullet)$ — the Lie algebra of regular vector fields on $C \setminus P_\bullet$ — on spaces of coinvariants that will be used later in §6 and §7. The residue theorem plays a role in the proof of Lemma 4.2, as in the classical case.

4.1. **The Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$.** Given a stable $n$-pointed curve $(C, P_\bullet)$ and a conformal vertex algebra $V$, the vector space

$$\mathscr{L}_{C\setminus P_\bullet}(V) := H^0\left(C \setminus P_\bullet, \mathscr{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right) \tag{17}$$

is a Lie algebra (as in the case $C$ smooth proved in [35, Cor. 19.4.14]). Here $\mathscr{V}_C$ is the vertex algebra sheaf on $C$ from §3.2.1, and $\nabla$ its flat connection from (16).

4.2. **The Lie algebra $\mathscr{L}_P(V)$.** For a stable pointed curve $(C, P)$, let $D_P^\times$ be the punctured disk about $P$ on $C$. That is, $D_P^\times = \mathrm{Spec}\,\mathscr{K}_P$, where $\mathscr{K}_P$ is the field of fractions of $\widehat{\mathscr{O}}_P$, the completed local ring at $P$. Given a conformal vertex algebra $V$, let $\mathscr{V}_C$ be the vertex algebra sheaf on $C$ from §3.2.1, and $\nabla$ its flat connection from (16). Define

$$\mathscr{L}_P(V) := H^0\left(D_P^\times, \mathscr{V}_C \otimes \omega_C/\mathrm{Im}\nabla\right). \tag{18}$$

The vector space $\mathscr{L}_P(V)$ forms a Lie algebra isomorphic to $\mathfrak{L}(V)$ [35, Cor. 19.4.14], and can be thought of as a coordinate-independent realization of $\mathfrak{L}(V)$ based at $P$. Moreover, the action of $\mathfrak{L}(V)$ on a $V$-module $M$ extends to an action of the Lie algebra $\mathscr{L}_P(V)$ on $M_{P,\tau}$ (see (9)), for any non-zero 1-jet $\tau$ at $P$ [35, Thm 7.3.10]. Similarly, the following lemma shows that the action of $\mathfrak{L}(V)$ on M induces a coordinate-independent action of $\mathscr{L}_P(V)$ on $M_P$ (§3.2.2).

**Lemma 4.1.** *There exists an anti-homomorphism of Lie algebras*

$$\alpha_M \colon \mathscr{L}_P(V) \to \mathrm{End}(M_P).$$

When the action of $L_0$ on $M$ has integral eigenvalues, Lemma 4.1 is the content of [35, §7.3.7]. In general, the proof proceeds as in [35, §6.5.4]. We present it here for completeness.

*Proof.* One first shows that it is possible to define a coordinate-free version of the map $Y^M(-, z)$. This is a map $\mathcal{Y}_P^M$ which assigns to every section in $H^0(D_P^\times, \mathscr{V}_C)$ an element of $\mathrm{End}(\mathscr{M}_{D_P^\times})$. Let us fix elements $s \in H^0(D_P^\times, \mathscr{V}_C)$, $v \in M_P$, and $\phi \in M_P^\vee$. To define $\mathcal{Y}_P^M$, it is enough to assign an element

$$\langle \phi, \mathcal{Y}_P^M(s)v\rangle \in \mathscr{O}_C(D_P^\times)$$

to every $s$, $v$, and $\phi$ as above. Once we choose a local coordinate $z$ at $P$, we can identify $s$ with an element $A_z$ of $V[\![z]\!]$, the element $v$ with an element $v_z$ of $M$ and $\phi$ with an element $\phi_z$ of $M^\vee$. Since the map $\mathcal{Y}_P^M$ must be $\mathscr{O}_{D_P}$-linear, we can further assume that $A_z \in V$, and that it is homogeneous. We then define

$$\langle \phi, \mathcal{Y}_P^M(s)v\rangle := \phi_z\left(Y^M(A_z, z)v_z\right).$$

We have to show that this is independent of the choice of the coordinate $z$. It has already been shown in [35, §7.3.10] that this map is $(\mathrm{Aut}_+\mathcal{O})$-invariant, so we only show that it is invariant under the change of coordinate $z \to w := az$ for $a \in \mathbb{C}^\times$, i.e., we are left to prove the equality

$$\phi_z\left(Y^M(A_z, z)v_z\right) = \phi_w\left(Y^M(A_w, w)v_w\right). \tag{19}$$

We need to compare $A_z$, $v_z$, and $\phi_z$ with $A_w$, $v_w$, and $\phi_w$. Unraveling the action of $\mathbb{C}^\times$ on $V$ and $M$ coming from the change of coordinate as described in (15), we see that

$$A_w = a^{-1} \cdot A_z = a^{\deg A_z} A_z, \qquad v_w = a^{-1} \cdot v_z = a^{\deg v_z} v_z,$$

where we assume that $v_z$ is homogeneous for simplicity. Finally, the element $\phi_w$ is just the composition of $\phi_z$ with the action of $a$ on $M$. We can then rewrite the right-hand side of equation (19) as

$$(\phi_z \circ a)\left(Y^M(a^{\deg A_z} A_z, az) a^{\deg v} v_z\right)$$

so it is enough to show that

$$Y^M(A_z, z) v_z = a \cdot \left(Y^M(a^{\deg A_z} A_z, az) a^{\deg v} v_z\right) \quad \text{in } M[\![z, z^{-1}]\!].$$

One has

$$\begin{aligned} a \cdot \left((a^{\deg A_z} A_z)_{(i)} a^{\deg v} v (az)^{-i-1}\right) &= a^{\deg A_z + \deg v_z - i - 1} a \cdot \left((A_z)_{(i)} v_z\right) z^{-i-1} \\ &= a^{\deg A_z + \deg v_z - i - 1 - \deg (A_z)_{(i)}(v)} (A_z)_{(i)} v_z z^{-i-1} = (A_z)_{(i)} v_z z^{-i-1}. \end{aligned}$$

The last equality follows from (2). We conclude that $\mathcal{Y}_P^M$ is well defined. In fact this is a particular case of a more general result proved in [46]. Using the residue theorem and an argument similar to [35, §6.5.8], we induce from $\mathcal{Y}_P^M$ the dual map

$$\left(\mathcal{Y}_P^M\right)^\vee : H^0(D_P^\times, \mathscr{V} \otimes \omega_C) \to \operatorname{End}(M_P).$$

It remains to show that this factors through a map

$$\alpha_M \colon H^0(D_P^\times, \mathscr{V} \otimes \omega_C / \operatorname{Im}\nabla) \to \operatorname{End}(M_P).$$

This can be checked by choosing a local coordinate at $P$ as in [35, §7.3.7]. □

4.3. **Coinvariants.** Fix a stable $n$-pointed curve $(C, P_\bullet)$ with marked points $P_\bullet = (P_1, \dots, P_n)$. One has a Lie algebra homomorphism by restricting sections

$$\mathscr{L}_{C\setminus P_\bullet}(V) \to \oplus_{i=1}^n \mathscr{L}_{P_i}(V), \quad \mu \mapsto (\mu_{P_1}, \dots, \mu_{P_n}). \tag{20}$$

This is as in the case $C$ smooth proved in [35, Cor. 19.4.14].

We assume below that each irreducible component of $C$ contains a marked point; in particular, $C \setminus P_\bullet$ is affine.

Fix $\tau_\bullet = (\tau_1, \dots, \tau_n)$, where each $\tau_i$ is a non-zero 1-jet at $P_i$. Given an $n$-tuple of $V$-modules $M^\bullet := (M^1, \dots, M^n)$, the *space of coinvariants* of $M^\bullet$ at $(C, P_\bullet, \tau_\bullet)$ is defined as the quotient

$$\mathbb{V}^J(V; M^\bullet)_{(C,P_\bullet,\tau_\bullet)} := \otimes_{i=1}^n M^i_{P_i,\tau_i} \Big/ \mathscr{L}_{C\setminus P_\bullet}(V) \cdot \left(\otimes_{i=1}^n M^i_{P_i,\tau_i}\right). \tag{21}$$

Here, $M^i_{P_i,\tau_i}$ is the coordinate-independent realization of $M^i$ assigned at $(P_i, \tau_i)$ as in (9). The action of $\mathscr{L}_{C\setminus P_\bullet}(V)$ is the restriction of the action of $\oplus_{i=1}^n \mathscr{L}_{P_i}(V)$ via (20):

$$\mu \cdot (A_1 \otimes \cdots \otimes A_n) = \sum_{i=1}^n A_1 \otimes \cdots \otimes \alpha_{M^i}(\mu_{P_i}) \cdot A_i \otimes \cdots \otimes A_n,$$

for $A_i \in M^i_{P_i,\tau_i}$ and $\mu \in \mathscr{L}_{C\setminus P_\bullet}(V)$.

Similarly, the *space of coinvariants* of $M^\bullet$ at $(C, P_\bullet)$ is defined as the quotient

$$\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)} := \otimes_{i=1}^n M^i_{P_i} \Big/ \mathscr{L}_{C\setminus P_\bullet}(V) \cdot \left(\otimes_{i=1}^n M^i_{P_i}\right). \tag{22}$$

Here, $M^i_{P_i}$ is the coordinate-independent realization of $M^i$ assigned at $P_i$ as in §3.2.2.

4.4. **The vertex algebra $\mathrm{Vir}_c$ and triviality of the action of $\mathscr{T}_C(C \setminus P_\bullet)$ on coinvariants.** For $c \in \mathbb{C}$, let $U(\mathrm{Vir})$ be the universal enveloping algebra of the Virasoro algebra Vir, and define

$$\mathrm{Vir}_c = \mathrm{Ind}^{\mathrm{Vir}}_{\mathrm{Der}\,\mathcal{O}\oplus\mathbb{C}K}\,\mathbb{C}_c = U(\mathrm{Vir}) \otimes_{U(\mathrm{Der}\,\mathcal{O}\oplus\mathbb{C}K)} \mathbb{C}_c,$$

where $\mathrm{Der}\,\mathcal{O}$ acts by zero and K acts as multiplication by $c$ on the one dimensional module $\mathbb{C}_c \simeq \mathbb{C}$. One has that $\mathrm{Vir}_c$ has central charge $c$ as a module over the Virasoro algebra, and has the structure of a conformal vertex algebra (see [35, §2.5.6] for this and more details about $\mathrm{Vir}_c$).

Let $\mathscr{T}_C(C \setminus P_\bullet)$ be the Lie algebra of regular vector fields on $C \setminus P_\bullet$. For $V$-modules $M^\bullet = (M^1, \dots, M^n)$, we will see that the action of $\mathscr{L}_{C\setminus P_\bullet}(\mathrm{Vir}_c)$ extends an action of $\mathscr{T}_C(C \setminus P_\bullet)$ on coordinate-free realizations of $\otimes_i M^i$, hence:

**Lemma 4.2.** *$\mathscr{T}_C(C \setminus P_\bullet)$ acts trivially on $\mathbb{V}^J(\mathrm{Vir}_c; M^\bullet)_{(C,P_\bullet,\tau_\bullet)}$ and $\mathbb{V}(\mathrm{Vir}_c; M^\bullet)_{(C,P_\bullet)}$.*

*Proof.* When $n = 1$, and necessarily $C$ smooth to ensure that $C \setminus P$ is affine, one has $\mathscr{T}_C(C \setminus P) \hookrightarrow \mathscr{L}_P(\mathrm{Vir}_c)$ by [35, §19.6.5]. It follows that

$$\mathscr{T}_C(C \setminus P) \hookrightarrow \mathrm{Im}\left(\mathscr{L}_{C\setminus P}(\mathrm{Vir}_c) \to \mathscr{L}_P(\mathrm{Vir}_c)\right).$$

For more marked points, as in [35, §19.6.5], one has that

$$\mathrm{Im}\left(\mathscr{L}_{C\setminus P_\bullet}(\mathrm{Vir}_c) \to \oplus_{i=1}^n \mathscr{L}_{P_i}(\mathrm{Vir}_c)\right)$$

contains an extension of $\mathscr{T}_C(C \setminus P_\bullet)$ by the image of the map $\varphi$

$$\begin{array}{ccccc} H^0(C \setminus P_\bullet, \omega_C/d\mathscr{O}_C) & \xrightarrow{\varphi} & \oplus_{i=1}^n H^0(D^\times_{P_i}, \omega_C/d\mathscr{O}_C) & \longrightarrow & \oplus_{i=1}^n \mathscr{L}_{P_i}(\mathrm{Vir}_c) \\ \big\downarrow & & \wr\wr & & \\ \mathscr{L}_{C\setminus P_\bullet}(\mathrm{Vir}_c) & & \mathbb{C}^n & & \end{array}$$

inside the central space (isomorphic to) $\mathbb{C}^n$ of $\oplus_{i=1}^n \mathscr{L}_{P_i}(\mathrm{Vir}_c)$. The map $\varphi$ is obtained by restricting sections. The space of sections of $\omega_C/d\mathscr{O}_C$ on $D^\times_{P_i}$ is isomorphic to $\mathbb{C}$ for each $i$, as such sections are identified by their residue at $P_i$. The inclusions above are induced from $\omega_C \hookrightarrow \mathscr{V}ir_c \otimes \omega_C$, which gives an inclusion $\omega_C/d\mathscr{O}_C \hookrightarrow \mathscr{V}ir_c \otimes \omega_C/\mathrm{Im}\nabla$. Here, $\mathscr{V}ir_c$ is the sheaf $\mathscr{V}_C$ for $V = \mathrm{Vir}_c$.

As $C \setminus P_\bullet$ is affine, one has $H^0(C \setminus P_\bullet, \omega_C) \twoheadrightarrow H^0(C \setminus P_\bullet, \omega_C/d\mathscr{O}_C)$. By the residue theorem, the image of $\varphi$ consists of the hyperplane of points $(r_1, \dots, r_n)$ in $\mathbb{C}^n$ such that $r_1 + \cdots + r_n = 0$. As each central space $\mathbb{C} \subset \mathscr{L}_{P_i}(\mathrm{Vir}_c)$ is $\mathbb{C} \simeq \mathbb{C} \cdot |0\rangle_{[-1]}$, and $|0\rangle_{[-1]}$ acts as the identity, the image of $\varphi$ acts trivially. □

For any conformal vertex algebra $V$ of central charge $c$, there is a vertex algebra homomorphism $\mathrm{Vir}_c \to V$ [35, §3.4.5]. This gives a surjection $\mathbb{V}(\mathrm{Vir}_c; M^\bullet)_{(C,P_\bullet)} \twoheadrightarrow \mathbb{V}(V; M^\bullet)_{(C,P_\bullet)}$ and similarly for the coinvariants $\mathbb{V}^J(-; M^\bullet)_{(C,P_\bullet,\tau_\bullet)}$ [35, §10.2.2].

The following statement will be used in the proof of Theorem 6.1.

**Lemma 4.3.** *$\mathscr{T}_C(C \setminus P_\bullet)$ acts trivially on $\mathbb{V}^J(V; M^\bullet)_{(C,P_\bullet,\tau_\bullet)}$ and $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)}$.*

*Proof.* Equivalent to the surjection on coinvariants referred to above, one has inclusions

$$\mathscr{L}_{C\setminus P_\bullet}(\mathrm{Vir}_c) \cdot \left(\otimes_{i=1}^n M^i_{P_i}\right) \hookrightarrow \mathscr{L}_{C\setminus P_\bullet}(V) \cdot \left(\otimes_{i=1}^n M^i_{P_i}\right),$$
$$\mathscr{L}_{C\setminus P_\bullet}(\mathrm{Vir}_c) \cdot \left(\otimes_{i=1}^n M^i_{P_i,\tau_i}\right) \hookrightarrow \mathscr{L}_{C\setminus P_\bullet}(V) \cdot \left(\otimes_{i=1}^n M^i_{P_i,\tau_i}\right).$$

Therefore, Lemma 4.3 follows from Lemma 4.2. □

## 5. Sheaves of coinvariants

In this section we define the sheaves of coinvariants $\widehat{\mathbb{V}}_g(V;M^\bullet)$, $\mathbb{V}_g^J(V;M^\bullet)$, and $\mathbb{V}_g(V;M^\bullet)$ on the moduli spaces $\widehat{\mathcal{M}}_{g,n}$, $\overline{\mathcal{J}}^{1,\times}_{g,n}$, and $\overline{\mathcal{M}}_{g,n}$, respectively. We will sometimes drop the $g$ or the $V$ in the notation (or both), when the context makes it clear.

Working in families [35, §17.3.9], one defines a sheaf of Lie subalgebras $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ with fibers equal to the Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$ from (17). This is used to define the sheaf of coinvariants $\widehat{\mathbb{V}}_g(V;M^\bullet)$ on $\widehat{\mathcal{M}}_{g,n}$ (§5.2). In §5.3.1 we will see that $\widehat{\mathbb{V}}_g(V;M^\bullet)$ is an $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$-module, hence descends along the principal $(\mathrm{Aut}_+\mathcal{O})^n$-bundle $\widehat{\pi}\colon \widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}^{1,\times}_{g,n}$, allowing one to describe the sheaf $\mathbb{V}_g^J(V;M^\bullet)$ on $\overline{\mathcal{J}}^{1,\times}_{g,n}$ (Def. 2). A second descent in §5.3.2 gives the sheaf $\mathbb{V}_g(V;M^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ (Def. 3). The structure and relationships between the three sheaves of coinvariants are summarized in Figure 5.

$$\begin{tikzcd}
 & & \widehat{\mathbb{V}}_g(V;M^\bullet) \arrow[dl] \arrow[dr] \arrow[ddd] & & \\
 & \mathbb{V}_g^J(V;M^\bullet) \arrow[rr] \arrow[ddd] & & \mathbb{V}_g(V;M^\bullet) \arrow[ddd] & \\
\mathbb{V}^J(V;M^\bullet)_{(C,P_\bullet,\tau_\bullet)} \arrow[ur, bend left] \arrow[ddd] & & & & \mathbb{V}(V;M^\bullet)_{(C,P_\bullet)} \arrow[ul, bend right] \arrow[ddd] \\
 & & \widehat{\mathcal{M}}_{g,n} \arrow[dl] \arrow[dr] & & \\
 & \overline{\mathcal{J}}^{1,\times}_{g,n} \arrow[rr] & & \overline{\mathcal{M}}_{g,n} & \\
\mathrm{Spec}(\mathbb{C}) \arrow[ur, bend left, "{(C,P_\bullet,\tau_\bullet)}"] \arrow[rrrr, mapsto] & & & & \mathrm{Spec}(\mathbb{C}) \arrow[ul, bend right, "{(C,P_\bullet)}"']
\end{tikzcd}$$

Figure 5. The structure of sheaves of coinvariants.

5.1. **Sheaves of Lie algebras on $\overline{\mathcal{M}}_{g,n}$ and on $\widehat{\mathcal{M}}_{g,n}$.** We will first define a sheaf of Lie algebras $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ on $\overline{\mathcal{M}}_{g,n}$, which we can regard as a sheaf on $\widehat{\mathcal{M}}_{g,n}$ by choosing formal coordinates at the marked points (that is, by pulling $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ back to $\widehat{\mathcal{M}}_{g,n}$ along the projection $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

Identify $\pi_{n+1}\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ with the universal curve $\pi_{n+1}\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Let $\overline{\mathcal{E}}_{g,n}$ be the moduli space of stable $(n+1)$-pointed genus $g$ curves, together with a formal coordinate at the last marked point. The natural projection $\overline{\mathcal{E}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ realizes $\overline{\mathcal{E}}_{g,n}$ as a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on $\overline{\mathcal{C}}_{g,n}$.

The group $\mathrm{Aut}\,\mathcal{O}$ acts equivariantly on the trivial vector bundle $\overline{\mathcal{E}}_{g,n} \times V$ over $\overline{\mathcal{E}}_{g,n}$. The quotient of $\overline{\mathcal{E}}_{g,n} \times V$ modulo the action of $\mathrm{Aut}\,\mathcal{O}$ descends to a vector bundle

$$V_{\overline{\mathcal{C}}_{g,n}} := \overline{\mathcal{E}}_{g,n} \underset{\mathrm{Aut}\,\mathcal{O}}{\times} V$$

on $\overline{\mathcal{C}}_{g,n}$. Let $\mathcal{V}_{\overline{\mathcal{C}}_{g,n}}$ denote the sheaf of sections of $V_{\overline{\mathcal{C}}_{g,n}}$. As in §3.2.1, the trivial bundle $\overline{\mathcal{E}}_{g,n} \times V$ has a flat logarithmic connection along the fibers of $\pi_{n+1}\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. The connection is $(\mathrm{Aut}\,\mathcal{O})$-equivariant, hence descends to a flat logarithmic connection on $\mathcal{V}_{\overline{\mathcal{C}}_{g,n}}$ along the fibers of $\pi_{n+1}$:

$$\nabla\colon \mathcal{V}_{\overline{\mathcal{C}}_{g,n}} \to \mathcal{V}_{\overline{\mathcal{C}}_{g,n}} \otimes \omega_{\pi_{n+1}}.$$

Define

$$\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V) := (\pi_{n+1})_*\,\mathrm{coker}\left(\nabla|_{\overline{\mathcal{C}}_{g,n}\setminus\mathcal{P}_\bullet}\right). \tag{23}$$

Here $\mathcal{P}_\bullet$ denotes the union of the images of the $n$ sections $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ given by the marked points. From [35, Cor. 19.4.14], $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ is a sheaf of Lie algebras on $\overline{\mathcal{M}}_{g,n}$. Pulling back to $\widehat{\mathcal{M}}_{g,n}$, after a choice of coordinates, gives a sheaf that we still call $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ on $\widehat{\mathcal{M}}_{g,n}$.

Next, we consider the sheaf of Lie algebras

$$\mathcal{L}(V)^n := \mathfrak{L}(V)^{\oplus n} \mathbin{\widehat{\otimes}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}} \tag{24}$$

on $\widehat{\mathcal{M}}_{g,n}$. By restricting sections, one can define a homomorphism of sheaves of Lie algebras:

$$\varphi\colon \mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V) \to \mathcal{L}(V)^n.$$

5.2. **The sheaf $\widehat{\mathbb{V}}_g(V; M^\bullet)$ on $\widehat{\mathcal{M}}_{g,n}$.** Let $\mathcal{C} \to S$ be a family of stable pointed curves over a smooth base $S$ with $n$ sections $P_i\colon S \to \mathcal{C}$ and formal coordinates $t_i$. Assume that $\mathcal{C}\setminus P_\bullet(S)$ is affine (e.g., $S = \widehat{\mathcal{M}}_{g,n}$). Let $V$ be a conformal vertex algebra and let $M^\bullet = (M^1, \dots, M^n)$ be conformal $V$-modules. With notation as above, we set

**Definition 1.** The *sheaf of coinvariants* $\widehat{\mathbb{V}}(V; M^\bullet)_{(\mathcal{C}\to S, P_\bullet, t_\bullet)}$ on $S$ is the quasi-coherent sheaf

$$\widehat{\mathbb{V}}(V; M^\bullet)_{(\mathcal{C}\to S, P_\bullet, t_\bullet)} := \otimes_{i=1}^n M^i \otimes \mathcal{O}_S \Big/ \mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V) \cdot \left(\otimes_{i=1}^n M^i \otimes \mathcal{O}_S\right).$$

We can remove the condition that $\mathcal{C} \setminus P_\bullet(S)$ be affine as in the construction of coinvariants obtained from representations of affine Lie algebras, see e.g., [30], [55]. First we need the following property. Let $\mathcal{C} \to S$ be a family of stable curves as above. Consider additional sections $Q_i\colon S \to \mathcal{C}$, with $i = 1, \dots, m$, such that $(C_s, P_\bullet(s) \cup Q_\bullet(s))$ is a stable pointed curve, for all $s \in S$, with corresponding formal coordinates $r_i$, and $m$ copies $(V, \dots, V)$ of the $V$-module $V$.

**Proposition 5.1** (Propagation of vacua)**.** *One has a canonical isomorphism*

$$\widehat{\mathbb{V}}(V; M^\bullet)_{(\mathcal{C}\to S, P_\bullet, t_\bullet)} \simeq \widehat{\mathbb{V}}(V; M^\bullet \cup (V, \dots, V))_{(\mathcal{C}\to S, P_\bullet\cup Q_\bullet, t_\bullet \cup r_\bullet)}.$$

*These isomorphisms are compatible as $(Q_\bullet, r_\bullet)$ varies.*

When $S$ is a point, the above statement has been proved in [35, §10.3.1], generalizing the similar result from [67] for sheaves of coinvariants obtained from representations of affine Lie algebras. The proof in [35, §10.3.1] extends over an arbitrary base $S$.

Consider a family of stable pointed curves $\mathcal{C} \to S$ as above, but without assuming that $\mathcal{C}\setminus P_\bullet(S)$ be affine. After an étale base change, we can assume that the family of stable pointed curves $\mathcal{C} \to S$ has $m$ additional sections $Q_i\colon S \to \mathcal{C}$ such that $\mathcal{C} \setminus \{P_\bullet(S) \cup Q_\bullet(S)\}$ is affine. Let $t_\bullet$ and $r_\bullet$ be the formal coordinates at $P_\bullet(S)$ and $Q_\bullet(S)$. By Definition 1 we can define

$$\widehat{\mathbb{V}}(V; M^\bullet \cup (V, \dots, V))_{(\mathcal{C}\to S, P_\bullet\cup Q_\bullet, t_\bullet \cup r_\bullet)}$$

and thanks to Proposition 5.1 this is independent of the chosen sections. We then define

$$\widehat{\mathbb{V}}(V; M^\bullet)_{(\mathcal{C}\to S, P_\bullet, t_\bullet)}$$

as the sheaf associated to the presheaf which associates to each $U \subset S$ the module

$$\varinjlim_{(Q_\bullet, r_\bullet)} \widehat{\mathbb{V}}(V; M^\bullet \cup (V, \dots, V))_{(\mathcal{C}\to U, P_\bullet\cup Q_\bullet, t_\bullet \cup r_\bullet)},$$

where $(Q_\bullet, r_\bullet)$ runs over all pairwise disjoint sections of $\mathcal{C}_U \setminus P_\bullet(U) \to U$ such that $\mathcal{C}_U \setminus \{P_\bullet(U) \cup Q_\bullet(U)\}$ is affine over $U$.

As the construction of the sheaf of coinvariants commutes with base change, one obtains a sheaf of coinvariants on $\widehat{\mathcal{M}}_{g,n}$, denoted $\widehat{\mathbb{V}}_g(V; M^\bullet)$.

5.3. **Two descents along torsors.**

5.3.1. *The first descent to define* $\mathbb{V}^J(V;M^\bullet)$ *on* $J := \overline{\mathcal{J}}^{1,\times}_{g,n}$. The group $(\mathrm{Aut}_+\mathcal{O})^n$ acts by conjugation on $\mathfrak{L}(V)^{\oplus n}$. Together with the action of $(\mathrm{Aut}_+\mathcal{O})^n$ on $\widehat{\mathcal{M}}_{g,n}$, this gives rise to an equivariant action of $(\mathrm{Aut}_+\mathcal{O})^n$ on the sheaf $\mathcal{L}(V)^n$. We will need the following statement:

**Lemma 5.2.** *The subsheaf* $\varphi\left(\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)\right)$ *of* $\mathcal{L}(V)^n$ *is preserved by the action of* $(\mathrm{Aut}_+\mathcal{O})^n$.

More precisely, let $\mathscr{L}'_{C\setminus P_\bullet}(V)$ be the fiber of $\varphi\left(\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)\right)$ at $(C,P_\bullet,t_\bullet)$. One has

$$\mathscr{L}'_{C\setminus P_\bullet}(V) = \rho^{-1}\mathscr{L}'_{C\setminus P_\bullet}(V)\rho$$

for all $\rho\in(\mathrm{Aut}_+\mathcal{O})^n$. This is a special case of (29).

Let $\mathcal{C}\to S$ be a family of stable pointed curves over a smooth base $S$ with $n$ sections $P_i\colon S\to\mathcal{C}$ and non-zero 1-jets $\tau_i$. Assume that $\mathcal{C}\setminus P_\bullet(S)$ is affine (e.g., $S=\mathcal{J}^{1,\times}_{g,n}$, the locus of smooth curves in $\overline{\mathcal{J}}^{1,\times}_{g,n}$). Define the principal $(\mathrm{Aut}_+\mathcal{O})^n$-bundle $\widehat{S}\to S$ as the pull-back of $\widehat{\pi}\colon\widehat{\mathcal{M}}_{g,n}\to\overline{\mathcal{J}}^{1,\times}_{g,n}$ via the moduli map $S\to\overline{\mathcal{J}}^{1,\times}_{g,n}$.

Let $V$ be a conformal vertex algebra and let $M^\bullet=(M^1,\dots,M^n)$ be conformal $V$-modules. The action of $\mathrm{Der}_+\,\mathcal{O}=\mathrm{Lie}(\mathrm{Aut}_+\mathcal{O})$ on each $M^i$ can be exponentiated to an action of $\mathrm{Aut}_+\mathcal{O}$ on $M^i$.

The action of $\mathfrak{L}(V)$ on each module $M^i$ via $\alpha_{M_i}\colon\mathfrak{L}(V)\to M^i$, compatible with the action of $\mathrm{Aut}_+\mathcal{O}$ (see §1.7), induces an anti-homomorphism of Lie algebras

$$\begin{array}{rccl}\alpha_{\otimes_{i=1}^n M^i}\colon & \mathfrak{L}(V)^{\oplus n} & \to & \mathrm{End}(\otimes_{i=1}^n M^i)\\ & (u_1,\dots,u_n) & \mapsto & \sum_{i=1}^n \mathrm{Id}_{M^1}\otimes\cdots\otimes\alpha_{M^i}(u_i)\otimes\cdots\otimes\mathrm{Id}_{M^n}\end{array}\tag{25}$$

compatible with the action of $(\mathrm{Aut}_+\mathcal{O})^n$. This extends $\mathcal{O}_{\widehat{S}}$-linearly to an $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant action of $\mathcal{L}(V)^n=\mathfrak{L}(V)^n\,\widehat{\otimes}\,\mathcal{O}_{\widehat{S}}$ on $\otimes_{i=1}^n M^i\otimes\mathcal{O}_{\widehat{S}}$. That is, $\mathcal{L}(V)^n$ and $(\mathrm{Aut}_+\mathcal{O})^n$ act compatibly on $\otimes_{i=1}^n M^i\otimes\mathcal{O}_{\widehat{S}}$. The sheaf $\mathcal{L}(V)^n$ is $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant. By Lemma 5.2, the image of $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ in $\mathcal{L}(V)^n$ is preserved by the action of $(\mathrm{Aut}_+\mathcal{O})^n$. It follows that the quasi-coherent sheaf $\widehat{\mathbb{V}}_g(V;M^\bullet)|_{\widehat{S}}$ is an $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant $\mathcal{O}_{\widehat{S}}$-module, hence descends along the principal $(\mathrm{Aut}_+\mathcal{O})^n$-bundle $\widehat{\pi}\colon\widehat{S}\to S$. Namely:

**Definition 2.** The *sheaf of coinvariants* $\mathbb{V}^J(V;M^\bullet)_{(\mathcal{C}\to S,P_\bullet,\tau_\bullet)}$ on $S$ is the quasi-coherent sheaf

$$\mathbb{V}^J(V;M^\bullet)_{(\mathcal{C}\to S,P_\bullet,\tau_\bullet)} := \left(\widehat{\pi}_*\,\widehat{\mathbb{V}}_g(V;M^\bullet)|_{\widehat{S}}\right)^{(\mathrm{Aut}_+\mathcal{O})^n}$$

whose sections are the $(\mathrm{Aut}_+\mathcal{O})^n$-invariant sections of $\widehat{\mathbb{V}}_g(V;M^\bullet)|_{\widehat{S}}$. Similarly to §5.2, one can remove the hypothesis that $\mathcal{C}\setminus P_\bullet(S)$ is affine, and the construction can be extended to obtain a sheaf on $\overline{\mathcal{J}}^{1,\times}_{g,n}$, denoted $\mathbb{V}^J_g(V;M^\bullet)$.

The fiber of $\mathbb{V}^J_g(V;M^\bullet)$ over a point $(C,P_\bullet,\tau_\bullet)$ in $\overline{\mathcal{J}}^{1,\times}_{g,n}$ is canonically isomorphic to the space of coinvariants (21). The sheaf $\mathbb{V}^J(V;M^\bullet)$ comes with a twisted logarithmic $\mathcal{D}$-module structure, detailed in §6. This construction is the result of the localization functor transforming modules for the Harish-Chandra pair $(\mathfrak{L}(V)^n,(\mathrm{Aut}_+\mathcal{O})^n)$ into twisted logarithmic $\mathcal{D}$-modules.

5.3.2. *The second descent to define* $\mathbb{V}_g(V;M^\bullet)$ *on* $\overline{\mathcal{M}}_{g,n}$. The action of $(\mathbb{C}^\times)^n$ on $\otimes_i M^i$ obtained from its $\mathbb{Z}^n$-gradation as in (15) induces an action of $(\mathbb{C}^\times)^n$ on $\mathbb{V}^J_g(V;M^\bullet)$. Indeed, since $\mathbb{C}^\times$ is realized as the quotient $\mathrm{Aut}\,\mathcal{O}/\mathrm{Aut}_+\mathcal{O}$, it is enough to check that the action of the Lie algebra $\mathscr{L}_{C\setminus P}(V)$ on $\otimes_i M^i$ is in independent of the choice of coordinates at the fixed points. This follows from (29).

Let $\mathcal{C} \to S$ be a family of stable pointed curves over a smooth base $S$ with $n$ sections $P_i \colon S \to \mathcal{C}$, and assume that $\mathcal{C} \setminus P_\bullet(S)$ is affine (e.g., $S = \mathcal{M}_{g,n}$). Define the principal $(\mathbb{C}^\times)^n$-bundle $S^J \to S$ as the pull-back of $j \colon \overline{\mathcal{J}}^\times_{g,n} \to \overline{\mathcal{M}}_{g,n}$ via the moduli map $S \to \overline{\mathcal{M}}_{g,n}$.

**Definition 3.** The *sheaf of coinvariants* $\mathbb{V}(V; M^\bullet)_{(\mathcal{C}\to S, P_\bullet)}$ on $S$ is the quasi-coherent sheaf

$$\mathbb{V}(V; M^\bullet)_{(\mathcal{C}\to S, P_\bullet)} := \left( j_* \, \mathbb{V}^J_g(V; M^\bullet)|_{S^J} \right)^{(\mathbb{C}^\times)^n}$$

whose sections are the $(\mathbb{C}^\times)^n$-invariant sections of $\mathbb{V}^J_g(V; M^\bullet)|_{S^J}$. This can be extended as in §5.2 to obtain a sheaf on $\overline{\mathcal{M}}_{g,n}$, denoted $\mathbb{V}_g(V; M^\bullet)$.

The fiber of $\mathbb{V}_g(V; M^\bullet)$ over a point $(C, P_\bullet)$ in $\overline{\mathcal{M}}_{g,n}$ is canonically isomorphic to the space of coinvariants (22).

5.3.3. *Prior work on extensions to stable curves.* Sheaves of coinvariants were originally constructed from integrable modules at a fixed level $\ell$ over affine Lie algebras $\hat{\mathfrak{g}}$ associated to semi-simple Lie algebras $\mathfrak{g}$ on $\widehat{\mathcal{M}}_{0,n} = \widehat{\mathcal{M}}_{0,n} \setminus \overline{\mathcal{M}}_{0,n}$ in [66]. These were generalized in [67] to $\widehat{\mathcal{M}}_{g,n}$, and shown to be locally free of finite rank, with a number of other good properties including a projectively flat connection. In [65] they were shown to be coordinate free, defined on $\overline{\mathcal{M}}_{g,n}$, and as is done here, an action of an Atiyah algebra on the bundles was explicitly computed.

Prior to our work, and aside from aspects of [18] and [35], considered here for stable curves with singularities, there has been some interest in sheaves of conformal blocks using vertex algebras. In [68] sheaves defined by representations at level $\ell$ over the Heisenberg vertex algebra were shown to be isomorphic to the space of theta functions of order $\ell$ on the curve, identified under pullback along the Abel-Jacobi map, and shown to be of finite rank. Sheaves defined using *regular* vertex algebras defined on $\widehat{\mathcal{M}}_{0,n}$ (regular vertex algebras are conformal and satisfy additional properties including, but not limited to, $C_2$-cofiniteness, which guarantees that they have finitely many simple modules), were shown to be of finite rank, and carry a projectively flat connection, as well as satisfy other good properties in parallel to the classical case [60]. The authors [60] remark that their definition of conformal blocks agrees with that described in [18, §§0.9, 0.10, 4.7.2], although their construction appears very different than what we consider here. Conformal blocks defined over smooth pointed curves with coordinates using modules over $C_2$-cofinite vertex algebras were shown to be finite-dimensional in [60]. Recently sheaves of coinvariants defined by holomorphic vertex algebras (admitting only one $V$-module: $V$ itself) on $\overline{\mathcal{M}}_{g,1}$ have been applied to the study of the Schottky problem and the slope of the effective cone of moduli of curves in [25].

Connections between conformal blocks defined by lattice vertex algebras and theta functions are discussed in [35, §5].

## 6. The twisted logarithmic $\mathcal{D}$-module structure on $\mathbb{V}^J(V; M^\bullet)$

In this section we specify the twisted logarithmic $\mathcal{D}$-module structure of the quasi-coherent sheaf of coinvariants $\mathbb{V}^J_g(V; M^\bullet)$ (Def. 2). In particular $\mathbb{V}^J_g(V; M^\bullet)$ supports a projectively flat logarithmic connection. To do this, we determine the (logarithmic) Atiyah algebra acting on sheaves of coinvariants.

Let $\Lambda := \det \mathbf{R}\pi_* \, \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ be the determinant of cohomology of the Hodge bundle (see §6.2), $\Delta$ the divisor in $\overline{\mathcal{M}}_{g,n}$ of singular curves, and $\mathcal{A}_\Lambda$ the corresponding logarithmic Atiyah algebra. One has:

**Theorem 6.1.** *The logarithmic Atiyah algebra $\frac{c}{2}\mathcal{A}_\Lambda$ acts on $\mathbb{V}^J_g(V; M^\bullet)$.*

Here $c$ is the central charge of the vertex algebra $V$. For $c = 0$, this gives a logarithmic $\mathcal{D}$-module structure on $\mathbb{V}_g^J(V; M^\bullet)$ (§6.4). We review the definition of a logarithmic Atiyah algebra in §6.1. For coinvariants of affine Lie algebras, the above statement is in [67]; see also [15, §7.10].

6.1. **Logarithmic Atiyah algebras.** Let $S$ be a smooth scheme over $\mathbb{C}$, and $\Delta \subset S$ a normal crossing divisor. Following [22] and [65], a *logarithmic Atiyah algebra* is a Lie algebroid $\mathcal{A}$ (see §2.1) together with its *fundamental extension sequence*:

$$0 \to \mathcal{O}_S \xrightarrow{\iota_{\mathcal{A}}} \mathcal{A} \xrightarrow{p_{\mathcal{A}}} \mathcal{T}_S(-\log \Delta) \to 0.$$

Here and throughout, $\mathcal{T}_S(-\log \Delta)$ is the sheaf of vector fields on $S$ preserving $\Delta$. One defines sums via the Baer sum $\mathcal{A} + \mathcal{B} := \mathcal{A} \times_{\mathcal{T}_S(-\log \Delta)} \mathcal{B}/\{(\iota_{\mathcal{A}}(f), -\iota_{\mathcal{B}}(f)),\ \text{for } f \in \mathcal{O}_S\}$, with fundamental sequence given by $\iota_{\mathcal{A}+\mathcal{B}} = (\iota_{\mathcal{A}}, 0) = (0, \iota_{\mathcal{B}})$, and $p_{\mathcal{A}+\mathcal{B}} = p_{\mathcal{A}} = p_{\mathcal{B}}$, and scalar multiplications $\alpha\mathcal{A} = (\mathcal{O}_S \oplus \mathcal{A})/(\alpha, 1)\mathcal{O}_S$, with $i_{\alpha\mathcal{A}} = (\mathrm{id}, 0)$ and $p_{\alpha\mathcal{A}} = (0, p_{\mathcal{A}})$, for $\alpha \in \mathbb{C}$. When $m \in \mathbb{Z}_{>0}$, one has $m\mathcal{A} = \mathcal{A} + \cdots + \mathcal{A}$ (with m summands). When $\Delta = \emptyset$, the above recovers the classical case of Atiyah algebras. Restrictions of Atiyah algebras over the subsheaf $\mathcal{T}_S(-\log \Delta) \hookrightarrow \mathcal{T}_S$ are logarithmic Atiyah algebras.

We will focus on logarithmic Atiyah algebras arising from line bundles: for a line bundle $\mathcal{L}$ over a smooth scheme $S$ equipped with a normal crossing divisor $\Delta$, we denote by $\mathcal{A}_{\mathcal{L}}$ the logarithmic Atiyah algebra of first order differential operators acting on $\mathcal{L}$ which preserve $\Delta$. Choosing a local trivialization $\mathcal{L} \simeq \mathcal{O}_S$ on an open subset $U$ of $S$, the elements of $\mathcal{A}_{\mathcal{L}}(U)$ are $D + f$, where $D \in \mathcal{T}_S(-\log \Delta)(U)$ and $f \in \mathcal{O}_S(U)$. For $\alpha \in \mathbb{C}$, while the line bundle $\mathcal{L}^\alpha$ may not be well-defined, the logarithmic Atiyah algebra $\mathcal{A}_{\mathcal{L}^\alpha}$ is defined as $\mathcal{A}_{\mathcal{L}^\alpha} := \alpha\mathcal{A}_{\mathcal{L}}$.

The action of a logarithmic Atiyah algebra $\mathcal{A}$ on a quasi-coherent sheaf $\mathbb{V}$ over $S$ is an action of sections $\mu \in \mathcal{A}$ as first order differential operators $\Phi(\mu)$ on $\mathbb{V}$ such that: (i) the principal symbol of $\Phi(\mu)$ is $p_{\mathcal{A}}(\mu) \otimes \mathrm{id}_{\mathbb{V}}$; (ii) $\Phi(\iota_{\mathcal{A}}(1)) = 1$. In particular, for $\alpha \in \mathbb{C}$ and a line bundle $\mathcal{L}$ on $S$, an action of $\mathcal{A}_{\mathcal{L}^\alpha}$ is an action of $\mathcal{A}_{\mathcal{L}}$ with the difference that $\iota_{\mathcal{A}_{\mathcal{L}}}(1)$ acts as multiplication by $\alpha$.

A flat connection with logarithmic singularities on a quasi-coherent sheaf $\mathbb{V}$ over $S$ is defined as a map of sheaves of Lie algebras $\mathcal{T}_S(-\log \Delta) \to \mathrm{End}(\mathbb{V})$. Following [20] we call projectively flat connection with logarithmic singularities a map $\mathcal{T}_S(-\log \Delta) \to \mathrm{End}(\mathbb{V})/(\mathcal{O}_S \,\mathrm{Id}_{\mathbb{V}})$. This means that the action of $\mathcal{T}_S(-\log \Delta)$ on $\mathbb{V}$ is well defined only up to constant multiplication. From the fundamental sequence defining an Atiyah algebra $\mathcal{A}$, it follows that the action of $\mathcal{A}$ on a quasi-coherent sheaf $\mathbb{V}$ induces a projectively flat connection with logarithmic singularities on $\mathbb{V}$.

6.2. **The Hodge line bundle.** For $n > 0$, let $n\mathrm{Vir}$ be the quotient of the direct sum $\mathrm{Vir}^{\oplus n}$ of $n$ copies of the Virasoro algebra modulo the identification of the $n$ central elements in $\mathrm{Vir}^{\oplus n}$ of type $(0, \ldots, 0, K, 0, \ldots, 0)$. We denote by $\overline{K}$ the equivalence class of these elements. Then $n\mathrm{Vir}$ is a central extension

$$0 \to \mathfrak{gl}_1 \cdot \overline{K} \to n\mathrm{Vir} \to (\mathrm{Der}\,\mathcal{K})^n \to 0$$

with Lie bracket given by

$$[(L_{p_i})_{1\le i\le n}, (L_{q_i})_{1\le i\le n}] = ((p_i - q_i)L_{p_i+q_i})_{1\le i\le n} + \frac{\overline{K}}{12}\sum_{i=1}^{n}(p_i^3 - p_i)\delta_{p_i+q_i,0}.$$

Let $\mathcal{C} \to S$ be a family of stable pointed curves of genus $g$ over a smooth base $S$ with $n$ sections $P_i\colon S \to \mathcal{C}$ and formal coordinates $t_i$. These data give a moduli map $S \to \widehat{\mathcal{M}}_{g,n}$. Assume that $\mathcal{C} \setminus P_\bullet(S)$ is affine. There is a Lie algebra embedding $\mathrm{Ker}\, a \hookrightarrow n\mathrm{Vir}\,\widehat{\otimes}\,\mathcal{O}_S$ [35, §19.6.5], where $a$ is the anchor map from (4). The quotient

$$n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S\Big/\mathrm{Ker}\,a$$

is anti-isomorphic to an extension

$$0 \to \mathcal{O}_S \to \mathcal{A} \to \mathcal{T}_S(-\log\Delta) \to 0$$

and carries the structure of a logarithmic Lie algebroid (see §2.1). The following statement is an immediate consequence of the *Virasoro uniformization* for the line bundle $\Lambda$ [22], [50], [67].

**Proposition 6.2.** *The logarithmic Atiyah algebra $\mathcal{A}$ is isomorphic to $\frac{1}{2}\mathcal{A}_\Lambda$.*

*Proof.* The Lie algebra $n\mathrm{Vir}$ acts on the line bundle $\Lambda$ with central charge 2 (the case $n = 1$ is in [50]). This action extends to an anti-homomorphism of logarithmic Lie algebroids

$$n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S \to \mathcal{A}_\Lambda$$

such that $1 \in \mathcal{O}_S \hookrightarrow n\mathrm{Vir}\,\widehat{\otimes}\,\mathcal{O}_S$ is mapped to $\frac{1}{2} \in \mathcal{O}_S \hookrightarrow \mathcal{A}_\Lambda$. The subsheaf $\mathrm{Ker}\,a \hookrightarrow n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S$ — whose fiber at $(C, P_\bullet, t_\bullet)$ consists of the Lie algebra $\mathscr{T}_C(C \setminus P_\bullet)$ of regular vector fields on $C \setminus P_\bullet$ — acts trivially on $\Lambda$ (e.g., [15, (7.10.11)] shows that $\mathrm{Ker}\,a$ acts trivially on $\Lambda^{-1}$). This gives a map of Atiyah algebras $\varphi\colon \mathcal{A} \to \frac{1}{2}\mathcal{A}_\Lambda$. Since Atiyah algebras are in $\mathrm{Ext}^1\left(\mathcal{T}_S(-\log\Delta), \mathcal{O}_S\right)$, the Five Lemma implies that $\varphi$ is an isomorphism. □

6.3. **The action on the sheaf of coinvariants.** Before the proof of Theorem 6.1, we discuss some auxiliary results.

The Lie algebra $\mathfrak{L}(V)^n$ contains $\mathrm{Vir}^n$ as a Lie subalgebra. The adjoint representation of $\mathrm{Vir}^n$ on $\mathfrak{L}(V)^n$ given by $v \mapsto [\cdot, v]$, for $v \in \mathrm{Vir}^n$, factors through an action of $n\mathrm{Vir}$ on $\mathfrak{L}(V)^n$. The Lie algebroid $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ acts on $\mathcal{L}(V)^n$ as

$$(v \otimes f) * (u \otimes h) := [u, v] \otimes fh + u \otimes (a(v \otimes f)\cdot h)\,, \tag{26}$$

for $v \in n\mathrm{Vir}$, $u \in \mathfrak{L}(V)^n$, and local sections $f, h$ of $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$. (The action is given by an *anti-homomorphism* of sheaves of Lie algebras from $n\mathrm{Vir}\,\widehat{\otimes}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ to the sheaf of endomorphisms of $\mathcal{L}(V)^n$. Indeed, the map $v \mapsto [\cdot, v]$, for $v \in n\mathrm{Vir}$, and the anchor map $a$ are both anti-homomorphisms. Here, the anchor map $a$ applies to the image of $v \otimes f$ in $(\mathrm{Der}\,\mathcal{K}(\mathbb{C}))^n\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$.

**Lemma 6.3.** *The image of $\mathcal{L}_{C\setminus\mathcal{P}_\bullet}(V)$ in $\mathcal{L}(V)^n$ is preserved by the action of $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ from* (26).

*Proof.* The case $n = 1$ and zero central charge $c = 0$ is [35, Thm 17.3.11]. The general case is similar; we sketch here a proof.

Let $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ be the group ind-scheme satisfying $\mathrm{Lie}\,\widehat{(\mathrm{Aut}\,\mathcal{K})^n} = n\mathrm{Vir}$. The action of $(\mathrm{Aut}\,\mathcal{K})^n$ on $\widehat{\mathcal{M}}_{g,n}$ induces an action of $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ on $\widehat{\mathcal{M}}_{g,n}$. Moreover, by exponentiating the action of $n\mathrm{Vir}$, one also has an action of $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ on $V^n$. (The exponential of the action of $n\mathrm{Vir}$ on $V^n$ is well defined: this follows from the fact that the action of $L_p$ with $p > 0$ on $V$ has negative degree $-p$, and can be integrated since $V$ has gradation bounded from below; the action of $L_0$ on $V$ has integral eigenvalues, and thus integrates to a multiplication by a unit; finally, the action of $L_p$ with $p < 0$ integrates to an action of a nilpotent element; see [35, §7.2.1]). This gives rise to an action of $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ on $\mathfrak{L}(V)^n$ by conjugation. Thus, $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ acts on $\mathcal{L}(V)^n$ via

$$(u \otimes h)\cdot\rho := (\rho^{-1}u\rho) \otimes (h\cdot\rho), \tag{27}$$

for $u \otimes h$ in $\mathcal{L}(V)^n$ and $\rho$ in $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$.

The differential of this action is an action of $n\mathrm{Vir}\,\widehat{\otimes}_{\mathbb{C}}\, H^0(\widehat{\mathcal{M}}_{g,n}, \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}})$ on $\mathcal{L}(V)^n$ given by

$$(v \otimes f) * (u \otimes h) := [u, v] \otimes fh + u \otimes (\alpha(v \otimes f) \cdot h)\,, \tag{28}$$

for $u \otimes h$ in $\mathcal{L}(V)^n$ and $v \otimes f \in n\mathrm{Vir}\,\widehat{\otimes}_{\mathbb{C}}\, H^0(\widehat{\mathcal{M}}_{g,n}, \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}})$. The action of $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ on $\mathcal{L}(V)^n$ from (26) is the $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$-linear extension of the action (28) of $n\mathrm{Vir}$ on $\mathcal{L}(V)^n$. (Recall from Theorem 2.1 that the anchor map $a$ is the $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$-linear extension of the map $\alpha$.) Hence, to prove the statement, it is enough to show that the action (27) of $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ on $\mathcal{L}(V)^n$ preserves the image of $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ in $\mathcal{L}(V)^n$.

Using the action of $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$ on $\widehat{\mathcal{M}}_{g,n}$, denote by $(C^\rho, P^\rho_\bullet, t^\rho_\bullet)$ the image of a coordinatized $n$-pointed curve $(C, P_\bullet, t_\bullet)$ via the action of $\rho$ in $\widehat{(\mathrm{Aut}\,\mathcal{K})^n}$. Let $\mathscr{L}'_{C\setminus P}(V)$ and $\mathscr{L}'_{C^\rho\setminus P^\rho_\bullet}(V)$ be the fibers of the subsheaf

$$\mathrm{Im}\left(\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V) \to \mathcal{L}(V)^n\right) \subset \mathcal{L}(V)^n$$

on $(C, P_\bullet, t_\bullet)$ and $(C^\rho, P^\rho_\bullet, t^\rho_\bullet)$, respectively. It is enough to show that

$$\mathscr{L}'_{C^\rho\setminus P^\rho_\bullet}(V) = \rho^{-1}\mathscr{L}'_{C\setminus P}(V)\rho \tag{29}$$

for $\rho \in \widehat{\mathrm{Aut}}\,\mathcal{K}$. As in [35, §17.3.12], this follows from a generalization of a result of Y.-Z. Huang [46, Prop. 7.4.1], [35, §17.3.13]. □

Let

$$\alpha_{\otimes_{i=1}^n M^i}\colon \mathfrak{L}(V)^{\oplus n} \to \mathrm{End}(\otimes_{i=1}^n M^i)$$

be the anti-homomorphism of Lie algebras induced by the action of $\mathfrak{L}(V)$ on each module $M^i$ as in (25). This map restricts to an action of $\mathrm{Vir}^n$, and since the central elements $(0, \ldots, 0, K, 0, \ldots, 0)$ all act as multiplication by the central charge $c$, this action factors through an action of $n\mathrm{Vir}$ on $\otimes_{i=1}^n M^i$. The Lie algebroid $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ acts on $\otimes_{i=1}^n M^i \otimes_{\mathbb{C}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ as

$$(v \otimes f) \diamond (w \otimes h) := \left(\alpha_{\otimes_{i=1}^n M^i}(v) \cdot w\right) \otimes fh + w \otimes (a(v \otimes f) \cdot h) \tag{30}$$

for $v \in n\mathrm{Vir}$, $w \in \otimes_{i=1}^n M^i$, and local sections $f, h$ of $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$. Recall that the action of $\mathfrak{L}(V)^n$ on $\otimes_{i=1}^n M^i$ extends $\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$-linearly to an action of $\mathcal{L}(V)^n$ on $\otimes_{i=1}^n M^i \otimes_{\mathbb{C}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$, and this induces an action of $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ on $\otimes_{i=1}^n M^i \otimes_{\mathbb{C}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$.

**Lemma 6.4.** *The subsheaf*

$$\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V) \cdot \left(\otimes_{i=1}^n M^i \otimes_{\mathbb{C}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}\right) \tag{31}$$

*of* $\otimes_{i=1}^n M^i \otimes_{\mathbb{C}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ *is preserved by the action of* $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$ *from* (30).

*Proof.* We need to show that

$$(v \otimes f) \diamond [(u \otimes h) \cdot (w \otimes t)] \tag{32}$$

is in (31), for all local sections $v \otimes f \in n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$, $u \otimes h \in \mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$, and $w \otimes t \in \otimes_{i=1}^n M^i \otimes_{\mathbb{C}} \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$. In (32) the first action is as in (30), and the second one is

$$(u \otimes h) \cdot (w \otimes t) := \left(\alpha_{\otimes_{i=1}^n M^i}(u) \cdot w\right) \otimes ht.$$

Expanding, one verifies that (32) is equal to

$$[(v\otimes f)*(u\otimes h)]\cdot(w\otimes t)+(u\otimes h)\cdot\left((\alpha_{\otimes_{i=1}^n M^i}(v)\cdot w)\otimes ft\right).$$

To show that this is in (31), it is enough to show that $(v\otimes f)*(u\otimes h)$ is in the image of $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ in $\mathcal{L}(V)^n$, where the action is as in (26). This follows from Lemma 6.3. □

*Proof of Theorem 6.1.* Let $\mathcal{C}\to S$ be a family of stable pointed curves of genus $g$ over a smooth base $S$ with $n$ sections $P_i$ and formal coordinates $t_i$. These data give a moduli map $S\to\widehat{\mathcal{M}}_{g,n}$. From the construction of the sheaf of coinvariants, we can reduce to the case when $\mathcal{C}\setminus P_\bullet(S)$ is affine (e.g., $S=\widehat{\mathcal{M}}_{g,n}=\widehat{\mathcal{M}}_{g,n}\setminus\Delta$).

The Lie algebroid $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S$ acts on $\otimes_{i=1}^n M^i\otimes_{\mathbb{C}}\mathcal{O}_S$ as in (30). By Lemma 6.4, the subsheaf

$$\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)\cdot\left(\otimes_{i=1}^n M^i\otimes_{\mathbb{C}}\mathcal{O}_S\right)$$

is preserved by this action. Since the map $\alpha_{\otimes_{i=1}^n M^i}$ and the anchor map $a$ are compatible with the action of $(\mathrm{Aut}_+\mathcal{O})^n$ (as in §1.7 and §2.2.2), the action of $n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S$ on $\otimes_{i=1}^n M^i\otimes_{\mathbb{C}}\mathcal{O}_S$ from (30) is $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant.

By Lemma 4.3, the action of $\mathcal{L}_{\mathcal{C}\setminus\mathcal{P}_\bullet}(V)$ extends an action of $\mathrm{Ker}\,a\subset n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S$. It follows that (30) induces an $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant action of the Lie algebroid

$$n\mathrm{Vir}(\mathbb{C})\,\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S\Big/\mathrm{Ker}\,a,$$

hence an action of $\mathcal{A}$ on the sheaf $\widehat{\mathbb{V}}(M^\bullet)_{(\mathcal{C}\to S,P_\bullet,t_\bullet)}$. Equivalently, for some $a\in\mathbb{C}$, there exists an $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant action of the logarithmic Atiyah algebra $a\mathcal{A}$ on $\widehat{\mathbb{V}}(M^\bullet)_{(\mathcal{C}\to S,P_\bullet,t_\bullet)}$, which descends to an action of $a\mathcal{A}$ on the sheaf $\mathbb{V}^J(M^\bullet)_{(\mathcal{C}\to S,P_\bullet,\tau_\bullet)}$.

To determine $a$, note that the central element $\overline{K}\in n\mathrm{Vir}$ is the image of $(K,0,\dots,0)\in\mathrm{Vir}^n\hookrightarrow\mathfrak{L}(V)^n$, which acts via (25) as $c\cdot\mathrm{id}$, by definition of the central charge $c$. It follows that $\overline{K}$ acts as $c\cdot\mathrm{id}$, hence $a=c$. That is, the logarithmic Atiyah algebra $c\mathcal{A}$ acts on $\mathbb{V}^J(M^\bullet)_{(\mathcal{C}\to S,P_\bullet,\tau_\bullet)}$. The statement follows from Proposition 6.2. □

6.4. **The case $c=0$.** In the case of zero central charge $c=0$, the action in (30) induces an $(\mathrm{Aut}_+\mathcal{O})^n$-equivariant action of

$$(\mathrm{Der}\,\mathcal{K}(\mathbb{C}))^n\;\widehat{\otimes}_{\mathbb{C}}\,\mathcal{O}_S\Big/\mathrm{Ker}\,a\simeq\mathcal{T}_S(-\log\Delta)$$

on $\widehat{\mathbb{V}}(M^\bullet)_{(\mathcal{C}\to S,P_\bullet,t_\bullet)}$. This gives a logarithmic $\mathcal{D}$-module structure on $\mathbb{V}^J(M^\bullet)$ when $c=0$.

## 7. The twisted logarithmic $\mathcal{D}$-module structure on $\mathbb{V}(V;M^\bullet)$

In Theorem 7.1 we specify the twisted logarithmic $\mathcal{D}$-module structure on the quasi-coherent sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$ in case the modules $M^i$ are simple. In particular $\mathbb{V}_g(V;M^\bullet)$ carries a projectively flat logarithmic connection. The twisted $\mathcal{D}$-module structure on the sheaf of coinvariants on the moduli space of *smooth* pointed curves was first introduced in [35, §17.3.20]. Here we extend the description over families of *stable* pointed curves, and identify the twisted logarithmic $\mathcal{D}$-module by determining the precise logarithmic Atiyah algebra acting on sheaves of coinvariants. For sheaves of coinvariants of integrable representations of an affine Kac-Moody algebra, this statement was proved by Tsuchimoto [65].

To state the result: let $\Lambda:=\det\mathbf{R}\pi_*\,\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ be the determinant of cohomology of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$, and $\Psi_i=P_i^*(\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}})$ the cotangent line bundle on $\overline{\mathcal{M}}_{g,n}$ corresponding to the

$i$-th marked point. Let $\mathcal{A}_\Lambda$, $\mathcal{A}_{\Psi_i}$ be the corresponding logarithmic Atiyah algebras with respect to the divisor $\Delta$ in $\overline{\mathcal{M}}_{g,n}$ of singular curves (see §6.1). We show:

**Theorem 7.1.** *When each $V$-module $M^i$ is simple of conformal dimension $a_i$, the logarithmic Atiyah algebra*

$$\frac{c}{2}\mathcal{A}_\Lambda + \sum_{i=1}^{n} a_i \mathcal{A}_{\Psi_i} \tag{33}$$

*acts on the sheaf of coinvariants $\mathbb{V}_g(V; M^\bullet)$. In particular, $\mathbb{V}_g(V; M^\bullet)$ carries a projectively flat logarithmic connection.*

Here $V$ is a conformal vertex algebra with central charge $c$ (see §1.2). Recall that a simple $V$-module $M$ has conformal dimension $a$ if $L_0 v = (a + \deg v)v$, for homogeneous $v \in M$.

*Proof.* From Theorem 6.1 we know that $\frac{c}{2}\mathcal{A}_\Lambda$ acts on $\mathbb{V}^J(M^\bullet)$. This means that for any family of stable pointed curves $\mathcal{C} \to S$ over a smooth base $S$ with $n$ sections $P_i \colon S \to \mathcal{C}$ and $n$ non-zero 1-jets $\tau_i$, $\frac{c}{2}\mathcal{A}_\Lambda$ acts on $\mathbb{V}^J(M^\bullet)_{(\mathcal{C}\to S, P_\bullet, \tau_\bullet)}$. It is enough to check that an action of $\frac{c}{2}\mathcal{A}_\Lambda + \sum_{i=1}^n a_i \mathcal{A}_{\Psi_i}$ is naturally induced on $\mathbb{V}(M^\bullet)_{(\mathcal{C}\to S, P_\bullet)}$. Assume for simplicity that $n = 1$.

Observe that by definition $\overline{\mathcal{J}}^{\times}_{g,1}$ is contained in $\Psi$. Following [14, Theorem 5] we know that $-\mathcal{A}_\Psi$ is identified, as a vector bundle, with the sheaf of tangent vectors of $\overline{\mathcal{J}}^{\times}_{g,1}$ which are invariant under the action of $\mathbb{C}^\times$. It follows that $\mathbb{V}(M)_{(\mathcal{C}\to S, P)}$ is equipped with an action of the Lie algebroid $\frac{c}{2}\mathcal{A}_\Lambda - \mathcal{A}_\Psi$. In order to conclude, it is enough to prove that the image of 1 under the canonical map $\mathcal{O}_S \to -\mathcal{A}_\Psi$ acts on $\mathbb{V}(M)_{(\mathcal{C}\to S, P)}$ as multiplication by $-a$.

The computation of $a$ is local on $S$. Fix a point on the divisor $Z := \Psi \setminus \overline{\mathcal{J}}^{\times}_{g,1}$ and near this point we can locally assume that $\overline{\mathcal{J}}^{\times}_{g,1}$ is $\operatorname{Spec}(R[w, w^{-1}])$, where $w$ is the coordinate defining the divisor $Z$ in $\Psi$. Since $\mathbb{C}^\times$ acts on $w$ by multiplication, it follows that the vector fields which are $(\mathbb{C}^\times)$-equivariant are given by $\mathscr{T}_R \oplus wR\partial_w$. Locally the exact sequence defining $\mathcal{A}_\Psi$ is then given by

$$0 \to R \to \mathscr{T}_R \oplus wR\partial_w \to \mathscr{T}_R \to 0$$

where 1 goes to $w\partial_w = -L_0$. Following the description of $\mathscr{M}_\mathcal{C}$ in §3.2.3, we similarly identify the elements of $\mathbb{V}(M)_{(\mathcal{C}\to S, P)}$ as being represented by linear combinations of elements of the form $v \otimes w^{\deg v}$, where $v$ is a homogeneous element of $M$. Observe that since the action of $-\mathcal{A}_\psi$ is compatible with the action of $\mathcal{L}_{\mathcal{C}\setminus P}(V)$ on $M \otimes \mathcal{O}_S$, we can carry the computation on $(\mathbb{C}^\times)$-invariant sections of $M \otimes \mathcal{O}_S$. The action of $w\partial w$ on $\mathbb{V}(M)_{(\mathcal{C}\to S, P)}$ is described in (30). The computation

$$\begin{aligned} w\partial w(v \otimes w^{\deg v}) &= -L_0(v) \otimes w^{\deg v} + v \otimes w\partial_w w^{\deg v} \\ &= -(a + \deg v)v \otimes w^{\deg v} + v \otimes (\deg v) w^{\deg v} \\ &= -a(v \otimes w^{\deg v}) \end{aligned}$$

concludes the argument. □

7.1. **Remark.** While in this work we study the projectively flat connection $\nabla$ for bundles on $\overline{\mathcal{M}}_{g,n}$ defined using conformal vertex algebras along the lines of [67, 65, 20, 15], there are other approaches and further questions studied in the classical case, where conformal blocks are known to be isomorphic to generalized theta functions (see e.g., [44, 32, 34, 52]). For instance, the *geometric unitary conjecture* [40, 31, 41] was an explicit proposal of a unitary metric on the Verlinde bundles that would be (projectively) preserved by the connection $\nabla$ on $\mathcal{M}_{g,n}$. The geometric unitary conjecture for $\mathfrak{sl}_2$ was proved by Ramadas [62], and for arbitrary Lie algebras in genus zero by

Belkale [23]. It would be interesting to know if this picture could be extended to bundles defined by modules over conformal vertex algebras.

## 8. Chern classes of vector bundles $\mathbb{V}_g(V;M^\bullet)$ on $\mathcal{M}_{g,n}$

An expression for Chern classes of vector bundles on $\mathcal{M}_{g,n}$ of coinvariants of affine Lie algebras was given in [56]. One of the key ingredients used in [56] is the explicit description of the projectively flat logarithmic connection from [65]. In a similar fashion, the projectively flat logarithmic connection from Theorem 7.1 allows to compute Chern classes of vector bundles of coinvariants of vertex algebras on $\mathcal{M}_{g,n}$ to arrive at Corollary 8.1 below.

Let $V$ be a conformal vertex algebra of central charge $c$, and $M^1,\dots,M^n$ be simple conformal $V$-modules. Let $a_i$ be the conformal dimension of $M^i$, for each $i$, that is, $L_0(v)=(a_i+\deg v)\,v$, for homogeneous $v\in M^i$. In this section, we assume that the sheaf of coinvariants $\mathbb{V}_g(V;M^\bullet)$ is the sheaf of sections of a vector bundle of finite rank on $\mathcal{M}_{g,n}$. This assumption is known to be true in special cases, as described in §8.1.

We also assume that the central charge $c$ and the conformal dimensions $a_i$ are rational; this is satisfied for a rational vertex algebra V such that $\dim V/\langle A_{(-2)}B : A,B\in V\rangle<\infty$, after [29].

Let $\lambda := c_1(\Lambda)$, and $\psi_i := c_1(\Psi_i)$. From Theorem 7.1 with $S=\mathcal{M}_{g,n}$, the action of the Atiyah algebra (33) gives a projectively flat connection on the sheaf of coinvariants $\mathbb{V}_g(V;M^\bullet)$ on $\mathcal{M}_{g,n}$. This determines the Chern character of the restriction of $\mathbb{V}_g(V;M^\bullet)$ on $\mathcal{M}_{g,n}$.

**Corollary 8.1.** *When $\mathbb{V}_g(V;M^\bullet)$ has finite rank on $\mathcal{M}_{g,n}$ and $c,a_i\in\mathbb{Q}$, one has*

$$\mathrm{ch}\left(\mathbb{V}_g(V;M^\bullet)\right)=\mathrm{rank}\,\mathbb{V}_g(V;M^\bullet)\,\cdot\,\exp\left(\frac{c}{2}\,\lambda+\sum_{i=1}^n a_i\psi_i\right)\in H^*(\mathcal{M}_{g,n},\mathbb{Q}).$$

*Equivalently, the total Chern class is*

$$c\left(\mathbb{V}_g(V;M^\bullet)\right)=\left(1+\frac{c}{2}\,\lambda+\sum_{i=1}^n a_i\psi_i\right)^{\mathrm{rank}\,\mathbb{V}_g(V;M^\bullet)}\in H^*(\mathcal{M}_{g,n},\mathbb{Q}).$$

*Proof.* The statement follows from the general fact that a $\mathcal{A}_{L^{\otimes a}}$-module $E$ of finite rank, with $L$ a line bundle and $a\in\mathbb{Q}$, satisfies $c_1(E)=(\mathrm{rank}\,E)\cdot a\,c_1(L)$ (e.g., [56, Lemma 5]); moreover, the projectively flat connection implies that $\mathrm{ch}(E)=(\mathrm{rank}\,E)\cdot\exp(c_1(E)/\mathrm{rank}\,E)$ (e.g., [49, (2.3.3)]). □

8.1. **Remark.** From Theorem 7.1, the sheaf $\mathbb{V}_g(V;M^\bullet)$ on $\mathcal{M}_{g,n}$ is equipped with a projectively flat connection. It follows that when $\mathbb{V}_g(V;M^\bullet)$ is a sheaf of finite rank on $\mathcal{M}_{g,n}$, $\mathbb{V}_g(V;M^\bullet)$ is also locally free [64, §2.7].

It is natural to expect finite-dimensionality of spaces of coinvariants constructed from vertex algebras with finitely many simple modules, when there is an expectation that the factorization property will hold [35]. This has been checked in special cases: For integrable highest weight representations at level $\ell$ of affine Lie algebras [67]; representations at level $\ell$ of the Heisenberg vertex algebra [68]; highest weight representations of the so-called minimal series for the Virasoro algebra [19]. In the cases mentioned above, the vertex algebras are both rational and $C_2$-cofinite. Moreover, spaces of coinvariants associated to modules over $C_2$-cofinite vertex algebras have also been shown to be finite-dimensional in other more general contexts [5, 60]. While $C_2$-cofiniteness was conjectured to be equivalent to rationality [4], this was disproved in [6]. While slightly technical to define, $C_2$-cofiniteness is natural [9], and satisfied by many types of vertex algebras including a large class of simple $\mathcal{W}$-algebras containing all exceptional $\mathcal{W}$-algebras, and in particular the minimal series of principal $\mathcal{W}$-algebras discovered by Frenkel, Kac, and Wakimoto [36], proved to be $C_2$-cofinite by Arakawa [10] (see also [11]), vertex algebras associated to positive-definite even lattices [69, 1, 48], and a number of vertex algebras formed using the orbifold construction [2, 3, 59, 7].

## References


[1] T. Abe. Rationality of the vertex operator algebra $V_L+$ for a positive definite even lattice $L$. *Math. Z.*, 249(2):455–484, 2005. ↑30

[2] T. Abe. $C_2$-cofiniteness of the 2-cycle permutation orbifold models of minimal Virasoro vertex operator algebras. *Comm. Math. Phys.*, 303(3):825–844, 2011. ↑30

[3] T. Abe. $C_2$-cofiniteness of 2-cyclic permutation orbifold models. *Comm. Math. Phys.*, 317(2):425–445, 2013. ↑30

[4] T. Abe, G. Buhl, and C. Dong. Rationality, regularity, and $C_2$-cofiniteness. *Trans. Amer. Math. Soc.*, 356(8):3391–3402, 2004. ↑29

[5] T. Abe and K. Nagatomo. Finiteness of conformal blocks over compact Riemann surfaces. *Osaka J. Math.*, 40(2):375–391, 2003. ↑1, ↑29

[6] D. Adamović and A. Milas. On the triplet vertex algebra $\mathscr{W}(p)$. *Adv. Math.*, 217(6):2664–2699, 2008. ↑29

[7] C. Ai. $C_2$-cofiniteness of cyclic-orbifold vertex operator superalgebras. *Algebra Colloq.*, 24(2):315–322, 2017. ↑30

[8] V. Alexeev, A. Gibney, and D. Swinarski. Higher-level $\mathfrak{sl}_2$ conformal blocks divisors on $\overline{\mathcal{M}}_{0,n}$. *Proc. Edinb. Math. Soc. (2)*, 57(1):7–30, 2014. ↑2

[9] T. Arakawa. A remark on the $C_2$-cofiniteness condition on vertex algebras. *Math. Z.*, 270(1-2):559–575, 2012. ↑29

[10] T. Arakawa. Associated varieties of modules over Kac-Moody algebras and $C_2$-cofiniteness of $W$-algebras. *Int. Math. Res. Not. IMRN*, (22):11605–11666, 2015. ↑30

[11] T. Arakawa. Rationality of $W$-algebras: principal nilpotent cases. *Ann. of Math. (2)*, 182(2):565–604, 2015. ↑30

[12] M. Arap, A. Gibney, J. Stankewicz, and D. Swinarski. $\mathfrak{sl}_n$ level 1 conformal blocks divisors on $\overline{\mathcal{M}}_{0,n}$. *Int. Math. Res. Not. IMRN*, (7):1634–1680, 2012. ↑2

[13] E. Arbarello, C. De Concini, V. G. Kac, and C. Procesi. Moduli spaces of curves and representation theory. *Comm. Math. Phys.*, 117(1):1–36, 1988. ↑2, ↑7

[14] M. F. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85:181–207, 1957. ↑28

[15] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001. ↑3, ↑24, ↑25, ↑28

[16] A. Beauville and Y. Laszlo. Conformal blocks and generalized theta functions. *Comm. Math. Phys.*, 164(2):385–419, 1994. ↑2

[17] A. Beauville, Y. Laszlo, and C. Sorger. The Picard group of the moduli of $G$-bundles on a curve. *Compositio Math.*, 112(2):183–216, 1998. ↑2

[18] A. Beilinson and V. Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004. ↑1, ↑23

[19] A. Beilinson, B. Feigin, and B. Mazur. Introduction to algebraic field theory on curves. *Unpublished manuscript*, 1991. ↑15, ↑29

[20] A. Beilinson and D. Kazhdan. Flat projective connections. *Unpublished manuscript*, 1991. ↑24, ↑28

[21] A. Beilinson, A. and J. Bernstein. A proof of Jantzen conjectures. In *I. M. Gelfand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 1–50. Amer. Math. Soc., Providence, RI, 1993. ↑8, ↑15

[22] A. A. Beilinson and V. V. Schechtman. Determinant bundles and Virasoro algebras. *Comm. Math. Phys.*, 118(4):651–701, 1988. ↑2, ↑7, ↑24, ↑25

[23] P. Belkale. Unitarity of the KZ/Hitchin connection on conformal blocks in genus 0 for arbitrary Lie algebras. *J. Math. Pures Appl. (9)*, 98(4):367–389, 2012. ↑28

[24] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986. ↑1

[25] G. Codogni. Vertex algebras and teichmüller modular forms. *arxiv*, pages 1–31, 2019. ↑23

[26] C. Damiolini, A. Gibney, and N. Tarasca. On factorization of conformal blocks from vertex algebras. *In preparation.* ↑2

[27] C. Dong and J. Lepowsky. *Generalized vertex algebras and relative vertex operators*, volume 112 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1993. ↑5

[28] C. Dong, H. Li, and G. Mason. Twisted representations of vertex operator algebras. *Math. Ann.*, 310(3):571–600, 1998. ↑1

[29] C. Dong, H. Li, and G. Mason. Modular-invariance of trace functions in orbifold theory and generalized Moonshine. *Comm. Math. Phys.*, 214(1):1–56, 2000. ↑29

[30] N. Fakhruddin. Chern classes of conformal blocks. In *Compact moduli spaces and vector bundles*, volume 564 of *Contemp. Math.*, pages 145–176. Amer. Math. Soc., Providence, RI, 2012. ↑2, ↑21

[31] F. Falceto, K. Gawedzki, and A. Kupiainen. Scalar product of current blocks in wzw theory. *Phys. Lett. B*, 260(1-2):101–108, 1991. ↑28

[32] G. Faltings. Stable $G$-bundles and projective connections. *J. Algebraic Geom.*, 2(3):507–568, 1993. ↑28

[33] G. Faltings. A proof for the Verlinde formula. *J. Algebraic Geom.*, 3(2):347–374, 1994. ↑2

[34] G. Felder. The KZB equations on Riemann surfaces. In *Symétries quantiques (Les Houches, 1995)*, pages 687–725. North-Holland, Amsterdam, 1998. ↑28

[35] E. Frenkel and D. Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2004. ↑1, ↑2, ↑3, ↑5, ↑6, ↑7, ↑9, ↑11, ↑12, ↑14, ↑16, ↑17, ↑18, ↑19, ↑20, ↑21, ↑23, ↑24, ↑25, ↑26, ↑27, ↑29

[36] E. Frenkel, V. Kac, and M. Wakimoto. Characters and fusion rules for $W$-algebras via quantized Drinfel'd-Sokolov reduction. *Comm. Math. Phys.*, 147(2):295–328, 1992. ↑29

[37] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988. ↑1

[38] I. B. Frenkel, Y.-Z. Huang, and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.*, 104(494):viii+64, 1993. ↑5

[39] I. B. Frenkel, J. Lepowsky, and A. Meurman. A natural representation of the Fischer-Griess Monster with the modular function $J$ as character. *Proc. Nat. Acad. Sci. U.S.A.*, 81(10, , Phys. Sci.):3256–3260, 1984. ↑1

[40] K. Gawedzki. Lectures on conformal field theory. In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 727–805. Amer. Math. Soc., Providence, RI, 1999. ↑28

[41] K. Gawedzki and A. Kupiainen. SU(2) Chern-Simons theory at genus zero. *Comm. Math. Phys.*, 135(3):531–546, 1991. ↑28

[42] N. Giansiracusa and A. Gibney. The cone of type $A$, level 1, conformal blocks divisors. *Adv. Math.*, 231(2):798–814, 2012. ↑2

[43] A. Gibney, D. Jensen, H.-B. Moon, and D. Swinarski. Veronese quotient models of $\overline{\mathrm{M}}_{0,n}$ and conformal blocks. *Michigan Math. J.*, 62(4):721–751, 2013. ↑2

[44] N. J. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380, 1990. ↑28

[45] G. Höhn. Conformal designs based on vertex operator algebras. *Adv. Math.*, 217(5):2301–2335, 2008. ↑1

[46] Y.-Z. Huang. *Two-dimensional conformal geometry and vertex operator algebras*, volume 148 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997. ↑7, ↑18, ↑26

[47] Y.-Z. Huang. Vertex operator algebras, the Verlinde conjecture, and modular tensor categories. *Proc. Natl. Acad. Sci. USA*, 102(15):5352–5356, 2005. ↑1, ↑2

[48] P. Jitjankarn and G. Yamskulna. $C_2$-cofiniteness of the vertex algebra $V_L^+$ when $L$ is a nondegenerate even lattice. *Comm. Algebra*, 38(12):4404–4415, 2010. ↑30

[49] S. Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5. ↑29

[50] M. L. Kontsevich. The Virasoro algebra and Teichmüller spaces. *Funktsional. Anal. i Prilozhen.*, 21(2):78–79, 1987. ↑2, ↑7, ↑25

[51] S. Kumar, M. S. Narasimhan, and A. Ramanathan. Infinite Grassmannians and moduli spaces of $G$-bundles. *Math. Ann.*, 300(1):41–75, 1994. ↑2

[52] Y. Laszlo. Hitchin's and WZW connections are the same. *J. Differential Geom.*, 49(3):547–576, 1998. ↑28

[53] J. Lepowsky and H. Li. *Introduction to vertex operator algebras and their representations*, volume 227 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004. ↑5

[54] H.-S. Li. Local systems of vertex operators, vertex superalgebras and modules. *J. Pure Appl. Algebra*, 109(2):143–195, 1996. ↑5

[55] E. Looijenga. From WZW models to modular functors. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 427–466. Int. Press, Somerville, MA, 2013. ↑21

[56] A. Marian, D. Oprea, and R. Pandharipande. The first Chern class of the Verlinde bundles. In *String-Math 2012*, volume 90 of *Proc. Sympos. Pure Math.*, pages 87–111. Amer. Math. Soc., Providence, RI, 2015. ↑2, ↑29

[57] A. Marian, D. Oprea, R. Pandharipande, A. Pixton, and D. Zvonkine. The Chern character of the Verlinde bundle over $\overline{\mathcal{M}}_{g,n}$. *J. Reine Angew. Math.*, 732:147–163, 2017. ↑2

[58] M. Miyamoto. Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness. *Duke Math. J.*, 122(1):51–91, 2004. ↑1

[59] M. Miyamoto. A $\mathbb{Z}_3$-orbifold theory of lattice vertex operator algebra and $\mathbb{Z}_3$-orbifold constructions. In *Symmetries, integrable systems and representations*, volume 40 of *Springer Proc. Math. Stat.*, pages 319–344. Springer, Heidelberg, 2013. ↑30

[60] K. Nagatomo and A. Tsuchiya. Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line. *Duke Math. J.*, 128(3):393–471, 2005. ↑1, ↑23, ↑29

[61] C. Pauly. Espaces de modules de fibrés paraboliques et blocs conformes. *Duke Math. J.*, 84(1):217–235, 1996. ↑2

[62] T. R. Ramadas. The "Harder-Narasimhan trace" and unitarity of the KZ/Hitchin connection: genus 0. *Ann. of Math. (2)*, 169(1):1–39, 2009. ↑28

[63] G. Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004. ↑1

[64] C. Sorger. La formule de Verlinde. *Astérisque*, (237):Exp. No. 794, 3, 87–114, 1996. Séminaire Bourbaki, Vol. 1994/95. ↑29

[65] Y. Tsuchimoto. On the coordinate-free description of the conformal blocks. *J. Math. Kyoto Univ.*, 33(1):29–49, 1993. ↑1, ↑23, ↑24, ↑27, ↑28, ↑29

[66] A. Tsuchiya and Y. Kanie. Vertex operators in the conformal field theory on $\mathbf{P}^1$ and monodromy representations of the braid group. *Lett. Math. Phys.*, 13(4):303–312, 1987. ↑1, ↑23

[67] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989. ↑1, ↑2, ↑7, ↑21, ↑23, ↑24, ↑25, ↑28, ↑29

[68] K. Ueno. On conformal field theory. In *Vector bundles in algebraic geometry (Durham, 1993)*, volume 208 of *London Math. Soc. Lecture Note Ser.*, pages 283–345. Cambridge Univ. Press, Cambridge, 1995. ↑1, ↑23, ↑29

[69] G. Yamskulna. $C_2$-cofiniteness of the vertex operator algebra $V_L^+$ when $L$ is a rank one lattice. *Comm. Algebra*, 32(3):927–954, 2004. ↑30

[70] Y. Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996. ↑4

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