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Moduli spaces reveal how objects like varieties or schemes of a particular type behave in families. Facts, unreachable by other means, can often be proved about such structures by considering them as points in a moduli space. As they can possess rich combinatorial structure, moduli spaces potentially provide new tools for making arguments about the objects which they parametrize. For this reason, moduli spaces themselves can be valuable test varieties.

The moduli space $\overline{\mathcal{M}}_{g,n}$, parametrizing stable n-pointed curves of genus g, gives insight into the study of smooth curves and their degenerations, and is a prototype for moduli of higher dimensional varieties. Most beneficially, as curves arise in so many contexts, it is a meeting ground, fundamentally touching a number of fields of mathematics and mathematical physics.

A natural goal when studying a space, like the moduli space of curves, is to characterize the maps admitted by it. From this perspective, vector bundles are important, as their sections can be thought of as functions; and globally generated bundles are particularly meaningful, as their sections define morphisms. The identification of basepoint free loci as being characteristic classes of particular globally generated vector bundles, or some other geometric loci, can give valuable information about these morphisms.

In recent work, I have studied aspects of vector bundles on $\overline{\mathcal{M}}_{g,n}$ assembled from modules over affine Lie algebras. Originally constructed in [TUY89], on $\overline{\mathcal{M}}_{0,n}$, they are known to be globally generated [Fak09]. These Verlinde bundles, or vector bundles of conformal blocks, have fibers dual to generalized theta functions, and in some cases admit enumerative interpretations in terms of Gromov-Witten loci [BG18]. I have used these relationships to better understand the objects identified. In earlier work, I studied aspects the moduli space of curves, using Mori theory and tropical geometry.

Here I briefly describe some of my results, arranged according to the following themes:

(1) Enumerative and geometric interpretations of conformal blocks
(2) Families of Conformal blocks divisors
(3) Compactifications and generalizations of the moduli space of curves
(4) Tropical compactifications and generalizations of the F-Conjecture

1. Enumerative and geometric interpretations of conformal blocks

In positive genus and type A, vector spaces of conformal blocks, taken together, form a finitely generated graded ring whose Proj bears a natural relationship with (a family of) moduli stacks of vector bundles over such curves [BG]. On the other hand, in genus zero, and in type A, one can identify first Chern classes of the Verlinde bundles with certain Gromov-Witten loci [BG18]. These relationships have interesting consequences.

Gromov-Witten loci of smooth projective homogeneous varieties and Verlinde bundles. In recent work with P. Belkale [BG18], we study base point free loci on $\overline{\mathrm{M}}_{0,n}$ by making use of Gromov-Witten theory for smooth projective homogeneous varieties. In particular, we give intersection formulas to compute expressions for such classes, and work these out explicitly in the case of quadrics and projective spaces. We prove that in the simplest case, basepoint free Gromov-Witten Loci on $\overline{\mathrm{M}}_{0,n}$ defined from homogeneous varieties, are equivalent to conformal blocks divisors. Namely, first Chern classes of Verlinde bundles from affine Lie algebras in type A at level 1, are equal to divisors obtained from base point free Gromov-Witten loci from projective space. This somewhat surprising identification of these two very different sets of objects yields interesting dividends. For instance, we learn that the Gromov-Witten divisors on $\overline{\mathcal{M}}_{0,n}$ coming from projective space define a full dimensional subcone of the cone of nef divisors. On the other side, we get an enumerative interpretation for the first Chern classes of conformal blocks in type A (at level one) which generalizes Witten's Dictionary, an enumerative interpretation for ranks for conformal blocks in type A (at all levels).

Finite Generation of the determinant of cohomology and geometric interpretations of conformal blocks. For G a simple, simply connected complex linear algebraic group, and C a stable curve of arithmetic genus $g \geq 2$, $\mathrm{Bun}_{\mathrm{G}}(C)$ is the stack parameterizing principal G-bundles on C. Given a representation $\mathrm{G} \to \mathrm{GL}(V)$, one can associate the determinant of cohomology line bundle $\mathcal{D}(V)$ on $\mathrm{Bun}_{\mathrm{G}}(C)$. In [BG, Theorem 1.1], we show, for $\mathrm{G} = \mathrm{SL}(r)$, and the standard representation $\mathrm{SL}(r) \to \mathrm{GL}(V)$,

$$\mathcal{A}^C_{\bullet} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D}(V)^{\otimes m}) \text{ is finitely generated.}$$

The stacks $\mathrm{Bun}_{\mathrm{SL}(r)}(C)$ are not proper, and so one does not expect even the constituent vector spaces $\mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D}^{\otimes m})$ are finite dimensional for any m.

For smooth C, the moduli space $\mathrm{SU}_C(r)$ of semistable vector bundles of rank r on C with trivial determinant, is isomorphic to $\mathrm{Proj}(\mathcal{A}^C_{\bullet})$ [BL94, Fal94]. By varying C, one obtains a flat family over $\mathcal{M}_g$, and extensions to $\overline{\mathcal{M}}_g$ have been considered [Pan96, Sim94]. In [BG, Theorem 1.2], we give an alternative completion of the family to $\overline{\mathcal{M}}_g$, where all fibers are normal projective varieties, constructed using conformal blocks.

For smooth curves C, conformal blocks are also known to correspond to global sections of a line bundle on the moduli *space* of G-bundles (and parabolic bundles in case of marked points) [BL94, Fal94, KNR94, Pau96, LS97]. In [BG, Theorems 1.3 and 1.4], we show this is also true for singular curves in some cases. Namely, if the level is sufficiently divisible, we prove that $\mathrm{Proj}(\mathcal{A}^C_{\bullet})$ is generated in degree 1. In earlier work [BGK16], using intersection theory on $\overline{\mathcal{M}}_g$, we were able to see in some cases that this was not the case, so while $\mathrm{Proj}(\mathcal{A}^C_{\bullet})$ may naturally lie in a weighted projective space, fibers may not be interpreted at the global sections of a line bundle on a variety in projective space, answering a long open question.

2. Families of conformal blocks divisors

The set of effective cycles of codimension c that nonnegatively intersect effective cycles of dimension c on X form a cone called the nef cone, denoted $\mathrm{Nef}^c(X)$. Morphisms give elements of the cone of nef divisors, and while $\mathrm{Nef}^1(X)$ is often difficult to describe, even its *shape* and *location* with respect to the cone of effective divisors gives valuable information. For instance, the cone of nef divisors and closed cone of effective divisors on $\overline{\mathrm{M}}_g$ only touch at the origin [GKM02]. Consequently, there are no nontrivial morphisms (with connected fibers) from $\overline{\mathrm{M}}_g$ to any lower dimensional projective variety. In contrast, there is no understanding of the nontrivial fibrations on $\overline{\mathrm{M}}_{0,n}$, although it is conjectured that the cone of nef divisors is polyhedral.

Verlinde bundles on $\overline{\mathrm{M}}_{0,n}$ are known to be globally generated [Fak09], and so characteristic classes of these bundles are nef. In fact, they generate full dimensional subcones of $\mathrm{Nef}^c(X)$ for all

c [Fak09, GM16]. As there are so many such classes, it is natural to try to understand how they behave in families, and whether subcones generated by the conformal blocks cycles are polyhedral.

Vanishing and identities. Characteristic classes of vector bundles of conformal blocks are subject to a number of types of identities: In [BGM15, Fak09], new rank and level identities are shown to hold for first Chern classes at the critical level, and divisors are trivial above the critical level. In [BG], we show Chern characters in type A are quasi-polynomial. In [BGM16] we show under certain hypothesis, first Chern classes are subject to additive identities.

Questions of nonvanishing. Inspired by unexpected examples of trivial divisors, in [BGM16] we introduce the problem of finding necessary and sufficient conditions to determine when the first Chern class of a bundle is nontrivial. Necessary and sufficient conditions for nonvanishing are found for $\mathfrak{sl}_2$.

Full dimensional polyhedral subcones. In [GG12],we show the infinite set of type A, *level one* conformal blocks divisors on $\overline{\mathrm{M}}_{0,n}$ spans a finitely generated, full-dimensional subcone of the cone of nef divisors. In [Kaz14], my former postdoc A. Kazanova has shown that the infinite set of S_n invariant, *rank one*, conformal blocks divisors for $\mathfrak{sl}_n$ on $\overline{\mathrm{M}}_{0,n}$, which she identifies, spans a finitely generated, full-dimensional cone generated by level one divisors. In [Hob15], my student N. Hobson has shown that infinite set of *rank one*, conformal blocks divisors for $\mathfrak{sl}_2$ (and their generalizations) on $\overline{\mathrm{M}}_{0,n}$, which she identifies, spans a finitely generated, full-dimensional cone generated by level one divisors.

3. Birational models and generalizations of the moduli space of curves

New birational models. The study of alternative compactifications of moduli spaces of pointed curves has been important both in understanding families of curves, as well as giving an explicit description of the birational geometry (in the sense of Mori theory) of $\overline{\mathrm{M}}_{g,n}$. Conformal blocks have given new tools for providing modular interpretations of images of birational contractions. These new moduli spaces have been shown to include for example, cyclic covers [Fed11], and GIT quotients [GG12, Gia13, GJM13, GJMS13] generalizing Kapranov's compactifications of $\mathrm{M}_{0,n}$ [Kap93a, Kap93b]. Images have modular interpretations, parametrizing weighted points, supported on Veronese curves.

Conformal blocks and the Torelli map from $\overline{\mathrm{M}}_g$ to certain compactifications of A_g. While not known to be base point free generally, conformal blocks divisors can sometimes be used to study moduli of higher genus curves. In [Gib12], I identify one of the faces of the nef cone of $\overline{\mathrm{M}}_g$, consisting of semi-ample divisors, showing they are pullbacks of ample divisors along the Torelli map from $\overline{\mathrm{M}}_g$ to certain types of compactifications of A_g, the moduli space of Abelian varieties of dimension g.

Pointed trees of projective spaces. In [CGK09], Chen, Krashen and I construct a space $\mathrm{T}_{d,n}$, whose points correspond to n-pointed stable rooted trees of d-dimensional projective spaces, which for $d = 1$, are $(n + 1)$-pointed stable rational curves. Smooth and projective, with boundary a smooth normal crossings divisor, the space $\mathrm{T}_{d,n}$ has an inductive construction analogous to but differing from Keel's construction of $\overline{\mathrm{M}}_{0,n}$. We describe its Chow groups, Chow ring, Chow motive, and Poincaré polynomials, and use this information to answer a question of Fulton and MacPherson.

4. Tropical compactifications and a generalization of the F-Conjecture

The F-Conjecture asserts that a divisor on $\overline{\mathrm{M}}_{g,n}$ is nef if and only if it nonnegatively intersects one of a finite number of rational curves on $\overline{\mathrm{M}}_{g,n}$ called F-Curves. In particular, the F-Conjecture implies that $\mathrm{Nef}(\overline{\mathrm{M}}_{g,n})$ is polyhedral, the convex hull of a finite number of extremal rays.

Reduction to genus zero. In [GKM02], we proved that the F-Conjecture on $\overline{\mathrm{M}}_{0,g+n}$ implies the F-Conjecture on $\overline{\mathrm{M}}_{g,n}$. The conjecture holds for low g and n [KM13, GKM02, FG03, Gib09].

Reduction to log canonical. In [Gib09], I reduce the $F-$conjecture on $\overline{\mathrm{M}}_{g,n}$ to showing that certain divisors in $\overline{\mathrm{M}}_{0,N}$ for $N \leq g+n$ are equivalent to the sum of the canonical divisor plus an effective divisor supported on the boundary. As an application, I give numerical criteria, which if satisfied by a divisor D on $\overline{\mathrm{M}}_g$, show that D is nef. Using computer software, written by Krashen, this is used to verify the conjecture for $g \leq 24$.

Generalization to tropical compactifications. While generally not a toric variety, it is possible to embed $\overline{\mathrm{M}}_{0,n}$ into a noncomplete toric variety X_Δ, and in [GM10], we give equations for $\overline{\mathrm{M}}_{0,n}$ in the Cox ring of X_Δ. In [GM12], we study varieties X that, like $\overline{\mathrm{M}}_{0,n}$, can be embedded in a toric variety X_Δ, as a tropical compactification. We using X_Δ to define cones which are upper and lower bounds for $\mathrm{Nef}(X)$. For $X = \overline{\mathrm{M}}_{0,n}$, we show that our upper bound cone $U(\overline{\mathrm{M}}_{0,n})$, is the cone of divisors predicted by the F-Conjecture to be equal to $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. In other words, this result generalizes the F-Conjecture on $\overline{\mathrm{M}}_{0,n}$ to varieties that like $\overline{\mathrm{M}}_{0,n}$, are tropical compactifications.

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