Surveys on Recent Developments in Algebraic Geometry

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Contents

The algebraic geometry community has a tradition of running a research institute every ten years. This important conference convenes a majority of the practitioners in the world to overview the developments of the past decade and to outline the most fundamental and far-reaching problems for the next. Previous institutes have included Woods Hole (1964), Arcata (1974), Bowdoin (1985), Santa Cruz (1995) and Seattle (2005). This past Algebraic Geometry Summer Institute took place at the University of Utah, in Salt Lake City, on July 13–31, 2015.

Increasingly, algebraic geometry has become very diverse and technically demanding. It is daunting for graduate students and postdocs to master the techniques and wide-range of activities in the field. In response, the community has been running a "Bootcamp" in the week preceding the institute with the aim of familiarizing the participants with a broad-range of developments in algebraic geometry in an informal setting. In 2015, from July 6 to 10, the Bootcamp took place at the University of Utah, Salt Lake City. Following tradition, the 150 graduate student participants were mentored by 15 postdocs and young assistant professors, helping to create a vibrant and informal atmosphere and allowing young researchers to form support networks. The mentors introduced the graduate students and postdocs to exciting new developments in the Minimal Model Program, Hodge theory, perfectoid spaces, positive characteristic techniques, Boij-Söderberg theory, p-adic Hodge theory, Bridgeland stability, tropical geometry and many more topics at the cutting edge of the field.

Activities at the Bootcamp included morning lectures given by the mentors, followed by afternoon working sessions. In this volume, in an attempt to make these excellent expositions more widely available, we have collected 10 survey papers that grew out of the lectures. Each paper discusses a different subfield of algebraic geometry that has seen significant progress in the last decade. The papers preserve the informal tone of the lectures and strive to be accessible to graduate students who have basic familiarity with algebraic geometry. They also contain many illuminating examples and open problems. We expect these surveys will become invaluable resources, not only for graduate students and postdocs, but also senior researchers starting along new directions.

We now summarize the contents of this volume in greater detail.

Higher dimensional birational geometry. The Minimal Model Program was initiated in the 1980s by Mori as a way of extending classification theorems for surfaces to higher dimensional varieties. In the last decade, several of the central conjectures in the field have been resolved. The fundamental and influential work of Birkar, Cascini, Hacon and McKernan showed the existence of minimal models for varieties of general type, and proved the finite generation of the canonical ring. The consequences have ranged from the resolution of classical conjectures such as the Sarkisov program to the construction of new moduli spaces. The successes have also led to attempts at extending the theory to other contexts such as Kähler manifolds or characteristic p birational geometry. The paper *A snapshot of the Minimal Model Program* by Brian Lehmann describes the most important developments of the decade and gives a list of open problems and conjectures. The paper *Positive characteristic algebraic geometry* by Patakfalvi, Schwede and Tucker gives a detailed introduction to the use of the Frobenius morphism for defining birational and singularity invariants in characteristic p. This approach has led to tremendous advances including an extension of the Minimal Model Program to threefolds in characteristic $p > 5$. The paper contains many illustrative and instructional examples and exercises to familiarize the reader with the characteristic p techniques.

Progress on moduli spaces. Moduli spaces have been at the center of algebraic geometry and its applications to other areas of mathematics. Moduli spaces of curves, moduli spaces of abelian varieties and moduli spaces of sheaves appear in many guises in mathematics and have applications to number theory, combinatorics, mathematical physics and topology. In the last decade, there has been significant development in constructing new compactifications and new birational models of important moduli spaces. The developments were motivated in part by evolution in the Minimal Model Program and new techniques coming from derived categories and Bridgeland stability. As a result, our understanding of classical moduli spaces such as the moduli space of principally polarized abelian varieties and moduli spaces of Gieseker semistable sheaves on surfaces has vastly improved. The paper *The geometry of the moduli space of curves and abelian varieties* by Tommasi gives an overview of this progress. Tommasi's paper is notable for an accessible account of toroidal compactifications of the moduli spaces of abelian varieties. The paper *Birational geometry of moduli spaces of sheaves and Bridgeland stability* by Jack Huizenga gives a masterful introduction to the geometry of moduli spaces of sheaves on surfaces and Bridgeland stability conditions using the projective plane as a motivating example. In recent years, Bridgeland stability has revolutionized our understanding of the birational geometry of moduli spaces of sheaves on surfaces. Arcara, Bayer, Bertram, Coskun, Huizenga, Macrì and others have computed the ample and effective cones of moduli spaces of sheaves on certain surfaces and have given many geometric applications. Huizenga's paper also clearly illustrates the interactions between developments in the Minimal Model Program and moduli spaces.

New applications of moduli spaces and connections to dynamics. There has been significant expansion in the applications of moduli theory and in the interactions of moduli spaces with dynamics. Inspired by string theory, Gromov-Witten theory revolutionized classical enumerative geometry in the 1990s and early 2000s. In the last decade, the theory has matured and found new applications to the tautological ring of the moduli space of curves. Maulik, Nekrasov, Okounkov, Pandharipande conjectured the equivalence of various curve counting theories such as the Gromov-Witten and Donaldson-Thomas correspondence. Pandharipande, Pixton and others have resolved some of these conjectures. The paper *Gromov-Witten theory: From curve counts to string theory* by Clader, gives a succinct account of the vast advances that have taken place in Gromov-Witten theory in the last decade. Clader masterfully summarizes the correspondence between Gromov-Witten and Donaldson-Thomas theory and applications of Gromov-Witten theory to the tautological ring of the moduli space of curves. She concludes with a set of central open problems in the field. The paper *Teichmüller dynamics in the eyes of an algebraic geometer* by Chen, introduces algebraic geometers to recent developments in Teichmüller dynamics following the fundamental work of Eskin, Kontsevich, Mirzakhani and Zorich. In his beautifully illustrated survey, Chen makes many analytic and dynamical concepts accessible to algebraic geometers. Chen emphasizes applications of the theory to the geometry of the Deligne-Mumford moduli spaces of curves and degenerations of abelian differentials.

Rationality of varieties. The last few years have seen an explosion in the study of rationality of varieties. Voisin introduced a new deformation technique based on the Chow theoretic decomposition of the diagonal to show that very general quartic double solids are not stably rational. Her approach was extended by Colliot-Thélène and Pirutka, and applied by them and many others such as Totaro, Hassett and Tschinkel to resolve long standing questions of rationality and stable rationality. In addition, Kuznetsov and others have suggested using the derived category and orthogonal decompositions to obtain new invariants for rationality. The survey *Unramified cohomology, derived categories and rationality* by Auel and Bernardara, gives a comprehensive introduction to these novel ideas and explores the interconnections between the two developments.

Hodge theory and degenerations. Hodge theory carries subtle transcendental information about the geometry of complex varieties. In the last decade, new insights have allowed a better understanding of degenerations of Hodge structures and the corresponding degenerations of varieties. The paper *Degenerations of Hodge structure* by Robles is an introduction to recent developments in Hodge theory, most notably to the classification of certain degenerations of Hodge structure pioneered by Green, Griffiths, Kerr, Robles and others.

Syzygies and cohomology tables. Betti tables of resolutions and cohomology tables of vector bundles are fundamental objects of study in commutative algebra and algebraic geometry. Eisenbud and Schryer's solution of the Boij-Söderberg conjecture and the resulting description of the cones of these tables have reshaped the theory in the last decade. The paper *Questions about Boij-Söderberg theory* by Erman and Sam surveys the developments and poses many fascinating and accessible further open problems.

Homotopy methods in algebraic geometry. The motivic, or $\mathbb{A}^1$ homotopy theory, introduced by Morel and Voevodsky, lies at the heart of recent progress, such as the classification of vector bundles on smooth complex affine varieties by Asok and Fasel. The paper *A primer for unstable motivic homotopy theory* by Antieau and Elmanto, gives an accessible introduction to this technical theory. Many key examples, useful exercises, enticing open problems and extensive references make this paper an indispensable reference for beginning practitioners of the subject.

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