Geometric interpretations for the algebra of conformal blocks

Angela Gibney

The University of Georgia

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This talk is about recent work with Prakash Belkale and Anna Kazanova.

Conformal blocks

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They were originally constructed using integrable representations of simple affine Lie algebras by Tsuchiya, Ueno, Yamada (1989).

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are determined by:

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Laszlo, Sorger 1997;
Pauly 1996.

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$$\bigoplus_{m \in \mathbb{Z}_{>0}} V[m]^*_{(C,\vec{p})} \cong \bigoplus_{m \in \mathbb{Z}_{>0}} \mathrm{H}^0(X_{(C,\vec{p})}, L^{\otimes m}_{(C,\vec{p})}),$$

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for example, if

$$\mathbb{V}[m] = \mathbb{V}(\mathfrak{sl}_{r+1}, \{m\,\lambda_1, \ldots, m\,\lambda_n\}, m\,\ell),$$

then

$$m\,\lambda_i = \sum_{i=1}^{r} m\,c_i\,\omega_i.$$

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On the other hand, one can compute the dimensions of the vector spaces $V[m]$ using the Verlinde formula, and study $X_{(C,\vec{p})}$.

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Do such geometric interpretations hold for

$$\bigoplus_{m \in \mathbb{Z}_{>0}} V[m]^*_{(C,\vec{p})}$$

at SINGULAR curves C?

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Given a point $(C, \vec{p}) \in \overline{\mathcal{M}}_{g,n}$, where C is SINGULAR, and a vector space of conformal blocks $V_{(C,\vec{p})}$, can one always find a projective polarized pair $(X_{(C,\vec{p})}, L_{(C,\vec{p})})$ such that

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The answer is NO.

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The answer is NO.

Counter examples are given by vector bundles of conformal blocks.

Vector bundles of conformal blocks

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Fakhruddin (improved by Mukhopadhyay) (g=0), and Marian, Oprea, Pandharipande (g >0) have given formulas for $c_1(\mathbb{V})$.

Theorem (Belkale, G, Kazanova)

There are vector bundles of conformal blocks $\mathbb{V}$ *on* $\overline{\mathcal{M}}_{g,n}$, *and points* $(C, \vec{p}) \in \overline{\mathcal{M}}_{g,n}$, *for which there are no polarized varieties* $(X_{(C,\vec{p})}, L_{(C,\vec{p})})$ *such that*

$$\bigoplus_m V[m]^*_{(C,\vec{p})} \cong \bigoplus_m \mathrm{H}^0(X_{(C,\vec{p})}, L^{\otimes m}_{(C,\vec{p})})$$

holds.

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$$\bigoplus_{m \in \mathbb{Z}_{>0}} \mathbb{V}(\mathfrak{sl}_2, \emptyset, m)|_C^*, \quad \text{on } \overline{\mathcal{M}}_2$$

where C is the curve with a separating node.

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We did this in cases where if geometric interpretations were to hold, then we would have useful information about the polarized varieties

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For example, we found obstructions in case

$$\Delta(X_{(C,\vec{p})}, L_{(C,\vec{p})}) = 0.$$

Theorem (Belkale, G, Kazanova)

Suppose that $\mathbb{V}$ *has* Δ*-invariant zero rank scaling, and that for all* $(C, \vec{p})$*, there are polarized pairs* $(X_{(C,\vec{p})}, L_{(C,\vec{p})})$ *such that*

$$\bigoplus_{m \in \mathbb{Z}_{>0}} \mathbb{V}[m]|^{*}_{(C,\vec{p})} \cong \bigoplus_{m \in \mathbb{Z}_{>0}} \mathrm{H}^0(X_{(C,\vec{p})}, L^{\otimes m}_{(C,\vec{p})}).$$

Then for $m \geq 1$*,*

$$c_1(\mathbb{V}[m]) = \sum_{i=1}^{D} \alpha_i(m) c_1(\mathbb{V}[i]), \qquad (1)$$

the $\alpha_i(m)$ *are (explicitly given) polynomials in* m*, and* D *is the volume of* $\mathbb{V}$*.*

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we show Eq 2 fails by intersecting with F-curves.

The Δ-invariant of a projective variety

The Δ-invariant of a projective variety

Definition

Let X be an irreducible projective variety with ample line bundle L. The Δ-invariant of the pair (X, L) is

$$\Delta(X, L) = \dim(X) + L^{\dim(X)} - h^0(X, L).$$

If $\Delta(X, L) = 0$

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$\Delta(X, L) = 0 \implies X$ a proj. variety of minimal degree.

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(1) Suppose that every point $(C, \vec{p}) \in \overline{\mathcal{M}}_{g,n}$, there exists $(X_{(C,\vec{p})}, L_{(C,\vec{p})})$, such that

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$\Delta(X_{(C,\vec{p})}, L_{(C,\vec{p})}) = 0$ some $(C,\vec{p}) \in \mathcal{M}_{g,n}$,

$$\implies \Delta(X_{(C,\vec{p})}, L_{(C,\vec{p})}) = 0 \ \forall \ (C,\vec{p}) \in \overline{\mathcal{M}}_{g,n}.$$

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0. rational normal curves;
1. projective spaces;
2. quadric hypersurfaces;
3. rational normal scrolls;
4. Veronese surfaces; and
5. Cones over the above.

(3) The function

$$f(m) = \operatorname{rk} \mathbb{V}[m]$$

determines the Delta invariant

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Definition

$$\mathrm{Vol}\,\mathbb{V} = L^{\dim(X)}.$$

For example:

0. $\mathrm{rk}(\mathbb{V}[m]) = dm + 1$, then $\mathbb{V}$ has $(\mathbb{P}^1, \mathcal{O}(d))$-scaling;
1. $\mathrm{rk}(\mathbb{V}[m]) = \binom{d+m}{m}$, $\mathbb{V}$ has $(\mathbb{P}^d, \mathcal{O}(1))$-scaling;
2. $\mathrm{rk}(\mathbb{V}[m]) = 2\binom{m+d-1}{d} + \binom{m+d-1}{d-1}$, $\mathbb{V}$ has quadric hypersurface scaling;
3. $\mathrm{rk}(\mathbb{V}[m]) = (m+1)(1 + \frac{m(a+b)}{2})$, $\mathbb{V}$ has $(S(a,b), \mathcal{O}(1))$ scaling[1];
4. $\mathrm{rk}(\mathbb{V}[m]) = (m+1)(2m+1)$, $\mathbb{V}$ has Veronese surface scaling;

[1] there are more general scrolls $S(a_1, \ldots, a_d) = (\mathbb{P}(\mathcal{E}), \mathcal{O}(1))$

Definition

We say that $\mathbb{V}$ has Δ-invariant zero rank scaling if $\Delta(X_{(C,\vec{p})}, L_{(C,\vec{p})}) = 0$ for $(C, \vec{p}) \in \mathcal{M}_{g,n}$ such that

$$\mathbb{V}|^*_{(C,\vec{p})} \cong \mathrm{H}^0(X_{(C,\vec{p})}, L_{(C,\vec{p})}).$$

We checked many many examples, and found that the predicted scaling identities held except possibly on particular loci.

We have similar results for bundles that don't have Δ-invariant zero rank scaling.

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For example, pairs (X, L) isomorphic to Coble's Quartic Hypersurface in $\mathbb{P}^7$, and Coble's Cubic hypersurface in $\mathbb{P}^8$ (BGK), and more recently in work with Prakash we look at other examples as well.

We also have some results which give conditions that guarantee geometric interpretations do exist.

Theorem (BGK)

Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ *on* $\overline{\mathcal{M}}_{g,n}$*. If* $\mathrm{rk}(\mathbb{V}[m]) = \binom{m+d}{d}$ $\forall$ *m, then*

$$\binom{m+d}{d+1} c_1(\mathbb{V}) = c_1(\mathbb{V}[m]) \, + \mathbb{D}_m,$$

where $\mathbb{D}_m$ *is an effective Cartier divisor supported on the boundary of* $\overline{\mathcal{M}}_{g,n}$*. Geometric extensions hold if and only if* $D_m = 0$*.*

Our experimentation has led to some conjectures.

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Conjecture (Belkale, G, Kazanova)

Given a vector bundle of conformal blocks $\mathbb{V}$ *of type A on* $\overline{\mathcal{M}}_{g,n}$*, for every* $x = (C, \vec{p}) \in Z = \mathcal{M}^{rt}_{g,n} \cup \Delta^0_{irr}$*, there is a polarized pair* $(\mathcal{X}_x, \mathcal{L}_x)$ *such that*

$$\bigoplus_{m \in \mathbb{Z}_{>0}} \mathbb{V}|^*_x \cong \bigoplus_{m \in \mathbb{Z}_{>0}} \mathrm{H}^0(\mathcal{X}_x, \mathcal{L}^m_x).$$

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For $g = 0$, $Z = \overline{\mathcal{M}}_{0,n}$.

Modification

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Conjecture (Belkale, G)

Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$*, and let* $x = [C] \in \overline{\mathcal{M}}_g$*,* $g \geq 2$ *be any (closed) point corresponding to a nodal curve* C*. Then there is a projective variety* $\mathcal{X}_x$ *and a big (equivalent to an ample plus effective) line bundle* $\mathcal{L}_x$ *on* $\mathcal{X}_x$ *such that*

$$\bigoplus_{m \geq 0} \mathbb{V}[m]|_x^* \cong \bigoplus_{\widetilde{m} \geq 0} \mathrm{H}^0(\mathcal{X}_x, \mathcal{L}_x^{\widetilde{m}}),$$

where $\widetilde{m} = m$ *if* C *is an irreducible curve, and* $\widetilde{m} = \frac{m}{2}$ *otherwise. If* $r = 2$*, then* $\mathcal{L}_x$ *is ample.*

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Let $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ be any vector bundle of conformal blocks on $\overline{\mathcal{M}}_{g,n}$, and $(C, \vec{p}) \in \overline{\mathcal{M}}_{g,n}$ any point. There is an isomorphism of algebras

$$\bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{V}[m]|^{*}_{(C;\vec{p})} \cong \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{H}^0(\mathrm{Parbun}_G(C, \vec{p}), \mathcal{L}_G(C, \vec{p})^{\otimes m}),$$

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$\mathrm{Parbun}_G(C, \vec{p})$ *quasi-parabolic* G*-bundles on* C*,*

$\mathcal{L}_G(C, \vec{p})$ *natural line bundle.*

Thank you!