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Moduli spaces of curves occupy a distinguished position in algebraic geometry. As moduli spaces, they give insight into the study of smooth curves and their degenerations, and as special varieties, they have played a principal role as a prototype for moduli of higher dimensional varieties [KSB88, Ale02, HM06, HKT06, HKT09, CGK09]. We have recently seen revolutionary advances in the minimal model program [BCHM10], and moduli of curves provides one of the chief concrete, non-trivial settings where this nuanced theory has been exhibited, and explored [HH09, HH13, AFSvdW16, AFS16a, AFS16b].

Here I very briefly describe my own research about moduli spaces of curves, arranged along the following four general themes:

(1) **Moduli of curves from vector bundles of conformal blocks**
(2) **Vector bundles of conformal blocks from moduli of curves**
(3) **Generalizations of moduli spaces of curves and their properties**
(4) **Mori theory on the moduli space of curves**

I also briefly describe my newer work, with Prakash Belkale, where we show the section ring for the pair $(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D})$ is finitely generated, for $\mathcal{D}$ the determinant of cohomology line bundle.

The moduli space of curves, the nef cone, and vector bundles of conformal blocks

The space $\mathrm{M}_{g,n}$ is an irreducible, quasi-projective variety, whose points correspond to the set of equivalence classes of smooth curves of genus g, marked by n labeled points. Smooth curves degenerate to singular ones, and so $\mathrm{M}_{g,n}$ is not compact. By adding curves with at worst simple nodal singularities and finitely many automorphisms, one obtains a projective variety $\overline{\mathrm{M}}_{g,n}$ [DM69, KM76]. The stack $\overline{\mathcal{M}}_{g,n}$, parametrizing flat families of stable curves, reflects the geometry of $\overline{\mathrm{M}}_{g,n}$, while being in certain ways easier to study. Since $\overline{\mathrm{M}}_{0,n}$ is a fine moduli space, these points of view are equivalent in genus zero.

Given a projective variety X, the cone of nef divisors $\mathrm{Nef}(X)$, is an invariant that consists of effective divisors that nonnegatively intersect all curves on X. The nef cone is dual to the closed cone of effective curves (aka the Mori cone). If $f : X \to Y$ is a morphism from X to another projective variety Y, and if A is an ample divisor on Y, then f^*A is an element of $\mathrm{Nef}(X)$. While $\mathrm{Nef}(X)$ is often difficult to describe, even its *shape* and *location* with respect to the cone of effective divisors gives valuable information about morphisms admitted by the variety X. For example, we know that the nef and effective cones of $\overline{\mathrm{M}}_g$ only touch at the origin [GKM02]. This implies that there are no morphisms (with connected fibers) from $\overline{\mathrm{M}}_g$ to any lower dimensional projective variety, other than a point.

The stacks $\overline{\mathcal{M}}_{g,n}$ admit vector bundles of conformal blocks, constructed using affine Lie algebras [TK88, TUY89, Fak09]. In case g is zero, the vector bundles are globally generated, and their first Chern classes are base point free, giving elements of the nef cone. The interplay between intersection theory on $\overline{\mathcal{M}}_{g,n}$ and properties of these vector bundles, highlights nontrivial links between algebraic geometry and representation theory.

1. Moduli of curves from vector bundles of conformal blocks

Recent findings have shown that various aspects of $\overline{\mathcal{M}}_{g,n}$ reflect underlying geometric structures embodied by vector bundles of conformal blocks. Four such results are described below.

Extensions of families to the boundary. In [BG16, Theorem 1.2], we show there is a flat family over $\overline{\mathcal{M}}_g$ whose fibers over smooth curves C, are moduli spaces $\mathrm{SU}_C(r)$, parameterizing semistable vector bundles of rank r with trivial determinant on C. All the fibers of the family are normal, projective varieties, constructed using conformal blocks.

Identification of faces of the nef cone. First Chern classes of vector bundles of conformal blocks, which are base point free on $\overline{\mathrm{M}}_{0,n}$, have been shown to give rise to many known contraction maps [GJMS13, AGS14], including the tautological maps. These can also be used to study the nef cone for higher genus curves [Fak09]. For instance, in [Gib12], I identify one of the faces of the nef cone of $\overline{\mathrm{M}}_g$, showing it corresponds to extensions of the Torelli map from $\overline{\mathrm{M}}_g$ to certain types of compactifications of A_g, the moduli spaceof Abelian varieties of dimension g.

New birational models. The study of alternative compactifications of moduli spaces of pointed curves has been important both in understanding families of curves, as well as giving an explicit description of the birational geometry (in the sense of Mori theory) of $\overline{\mathrm{M}}_{g,n}$. Conformal blocks have given new tools for providing modular interpretations of images of birational contractions of $\overline{\mathrm{M}}_{0,n}$. These new moduli spaces have been shown to include for example, cyclic covers [Fed11], and GIT quotients [Gia13, GJM13, GJMS13] generalizing Kapranov's compactifications of $\mathrm{M}_{0,n}$ [Kap93a, Kap93b]. The latter giving a different (but overlapping) set of modular interpretations than described in the work of Smyth [Smy13], in that they parametrize "embedded curves", as opposed to abstract ones (cf. [GJM13, p.245]).

Finite generation of full-dimensional subcones of the Nef cone. In [GG12],we show the infinite set of type A, *level one* conformal blocks divisors on $\overline{\mathrm{M}}_{0,n}$ spans a finitely generated, full-dimensional subcone of the cone of nef divisors. In [Kaz14], my former postdoc A. Kazanova has shown that the infinite set of S_n invariant, *rank one*, conformal blocks divisors for $\mathfrak{sl}_n$ on $\overline{\mathrm{M}}_{0,n}$, which she identifies, spans a finitely generated, full-dimensional cone generated by level one divisors. In [Hob15], my student N. Hobson has shown that infinite set of *rank one*, conformal blocks divisors for $\mathfrak{sl}_2$ (and their generalizations) on $\overline{\mathrm{M}}_{0,n}$, which she identifies, spans a finitely generated, full-dimensional cone generated by level one divisors.

2. Vector bundles of conformal blocks from the moduli space of curves

Important features of vector spaces of conformal blocks have been discovered by the study of vector bundles of conformal blocks on moduli spaces of curves. Three types of such results are described next.

Geometric interpretations at points on the boundary. At smooth curves, conformal blocks correspond to generalized theta functions [BL94, Fal94, KNR94, Pau96, LS97], and in [BG16, Theorems 1.3 and 1.4], we show this is true for singular curves as well, in prescribed circumstances. As we show in [BGK16], using intersection theory on $\overline{\mathcal{M}}_g$, this does not hold in all situations.

Vanishing and identities. Characteristic classes of vector bundles of conformal blocks are subject to a number of types of identities: In [BGM15, Fak09], new rank and level identities are shown to hold for first Chern classes at the critical level, and divisors are trivial above the critical level. In [BG16], we show Chern characters in type A are quasi-polynomial. In [BGM16] we show under certain hypothesis, first Chern classes are subject to additive identities.

Questions of nonvanishing. Inspired by unexpected examples of trivial divisors, in [BGM16] we introduce the problem of finding necessary and sufficient conditions to determine when the first Chern class of a bundle is nontrivial. Necessary and sufficient conditions for nonvanishing are found for $\mathfrak{sl}_2$.

3. Generalizations of the moduli space of curves, and its properties

Pointed trees of projective spaces. In [CGK09], Chen, Krashen and I construct a space $\mathrm{T}_{d,n}$, whose points correspond to n-pointed stable rooted trees of d-dimensional projective spaces, which for $d=1$, are $(n+1)$-pointed stable rational curves. Smooth and projective, with boundary a smooth normal crossings divisor, the space $\mathrm{T}_{d,n}$ has an inductive construction analogous to but differing from Keel's construction of $\overline{\mathrm{M}}_{0,n}$. We describe its Chow groups, Chow ring, Chow motive, and Poincaré polynomials, and use this information to answer a question of Fulton and MacPherson.

Tropical compactifications and a generalization of the F-Conjecture. While generally not a toric variety, it is possible to embed $\overline{\mathrm{M}}_{0,n}$ into a noncomplete toric variety X_Δ, and in [GM10], we give equations for $\overline{\mathrm{M}}_{0,n}$ in the Cox ring of X_Δ. In [GM12], we study varieties X that, like $\overline{\mathrm{M}}_{0,n}$, can be embedded in a toric variety X_Δ, as a tropical compactification. We using X_Δ to define cones which are upper and lower bounds for $\mathrm{Nef}(X)$. For $X=\overline{\mathrm{M}}_{0,n}$, we show that our upper bound cone $U(\overline{\mathrm{M}}_{0,n})$, is the cone of divisors predicted by the F-Conjecture to be equal to $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. In other words, this result generalizes the F-Conjecture on $\overline{\mathrm{M}}_{0,n}$ to varieties that like $\overline{\mathrm{M}}_{0,n}$, are tropical compactifications.

4. Mori theory on the moduli space of curves

The F-Conjecture asserts that a divisor on $\overline{\mathrm{M}}_{g,n}$ is nef if and only if it nonnegatively intersects one of a finite number of rational curves on $\overline{\mathrm{M}}_{g,n}$ called F-Curves. In particular, the F-Conjecture implies that $\mathrm{Nef}(\overline{\mathrm{M}}_{g,n})$ is polyhedral, the convex hull of a finite number of extremal rays.

In [GKM02], we proved that the F-Conjecture on $\overline{\mathrm{M}}_{0,g+n}$ implies the F-Conjecture on $\overline{\mathrm{M}}_{g,n}$. The conjecture holds for low g and n [KM13, GKM02, FG03, Gib09]. In [Gib09], I reduce the $F-$conjecture on $\overline{\mathrm{M}}_{g,n}$ to showing that certain divisors in $\overline{\mathrm{M}}_{0,N}$ for $N \leq g+n$ are equivalent to the sum of the canonical divisor plus an effective divisor supported on the boundary. As an application, I give numerical criteria, which if satisfied by a divisor D on $\overline{\mathrm{M}}_g$, show that D is ample. Using a computer program, The Nef Wizard, written by Krashen, this is used to verify the conjecture for $g \leq 24$.

5. New direction: Vector bundles on curves

For G a simple, simply connected complex linear algebraic group, and C a stable curve of arithmetic genus $g \geq 2$, $\mathrm{Bun}_{\mathrm{G}}(C)$ is the stack parameterizing principal G-bundles on C. To any representation $\mathrm{G} \to \mathrm{Gl}(V)$, there corresponds a distinguished line bundle $\mathcal{D}(V)$ on $\mathrm{Bun}_{\mathrm{G}}(C)$, the determinant of cohomology line bundle.

In [BG16, Theorem 1.1], we show, when G is $\mathrm{SL}(r)$, and for the standard representation $\mathrm{SL}(r) \to \mathrm{Gl}(V)$, one has that the ring $\mathcal{A}^C_\bullet$

$$\bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D}(V)^m)$$

is finitely generated.

The stacks $\mathrm{Bun}_{\mathrm{SL}(r)}(C)$ are not proper, and so one does not expect that the constituent vector spaces $\mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D}^m)$ to be finite dimensional for any m, much less that their infinite sum would be so. The result is well known for smooth curves C, and in this case: $\mathrm{Proj}(\mathcal{A}^C_\bullet) \cong \mathrm{SU}_C(r)$, where $\mathrm{SU}_C(r)$ is the moduli space parameterizing semistable vector bundles of rank r with trivial determinant on C [BL94, Fal94]. These form a flat family over $\mathcal{M}_g$, and it is natural to ask if one can extend it to a family over Deligne and Mumford's compactification $\overline{\mathcal{M}}_g$ (This problem was discussed in Section 1: Using conformal blocks, we extend the family).

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