ON HIGHER CHERN CLASSES OF VECTOR BUNDLES OF CONFORMAL BLOCKS

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ABSTRACT. In this short note, with an aim towards giving examples of elements in the Pliant cone, we give a number of vanishing results and identities for higher Chern classes of vector bundles of conformal blocks on $\overline{\mathrm{M}}_{0,n}$. As an application, we give a full dimensional subcone of the Pliant cone whose generators are extremal in the cone of nef cycles.

1. INTRODUCTION

The pseudo-effective cone $\overline{\mathrm{Eff}}_m(\mathrm{X})$ is the closure of the cone generated by classes of m-dimensional subvarieties on a projective variety X. If X is smooth, then one can define higher codimension analogues of cones of nef divisors by taking $\mathrm{Nef}^m(\mathrm{X})$ to be dual to $\overline{\mathrm{Eff}}_m(\mathrm{X})$. Many properties held by these cones when $m = 1$ fail more generally [Pet09, Voi10, DELV10, FL14]. To more accurately capture the properties of cones of nef divisors, Fulger and Lehmann have introduced three sub-cones: the Pliant cone, the base-point free cone, and the universally pseudoeffective cone. The smallest of these; the Pliant cone $\mathrm{Pl}^m(\mathrm{X}) \subset \mathrm{Nef}^m(\mathrm{X})$ is the closure of the cone generated by monomials in Schur classes of globally generated vector bundles on X.

The stack $\overline{\mathcal{M}}_{g,n}$, parametrizing flat families of stable n-pointed curves of genus g, carries vector bundles $\mathbb{V}$, constructed using representation theory [TUY89]. When $g = 0$, the bundles are globally generated, and higher Chern classes give rise to classes in the pliant cone on the moduli *space* $\overline{\mathrm{M}}_{0,n}$.

Here we explain how a number of vanishing results and identities governing first Chern classes of vector bundles of conformal blocks on $\overline{\mathrm{M}}_{0,n}$, may be extended to higher Chern classes:

- in Theorem 3.1, using an expression for the total Chern character $\mathrm{Ch}(\mathbb{V})$ given in [MOP+14], we give an explicit formula for the m-th Chern class $\mathrm{c}_m(\mathbb{V})$ on $\overline{\mathrm{M}}_{0,n}$;
- in Section 4, give additive and critical level identities governing higher Chern classes; and
- in Section 5, we give criteria for higher Chern classes to be extremal in the nef cone.

Using Fakhruddin's basis for $\mathrm{A}^1(\overline{\mathrm{M}}_{0,n})$, in Section 6.1 we form a full dimensional sub-cone of the Pliant cone $\mathrm{Pl}^m(\overline{\mathrm{M}}_{0,n})$ consisting of extremal classes in $\mathrm{Nef}^m(\overline{\mathrm{M}}_{0,n})$. With a basis for $\mathrm{A}^1(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$, from [AGSS11] consisting of classes that span extremal rays of the S_n-invariant cone of nef divisors, in Section 6.2, we construct a full dimensional sub-cone of the S_n-invariant Pliant cone $\mathrm{Pl}^m(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$. As with the generators of these subcone of $A^m(\overline{\mathrm{M}}_{0,n})$, and $A^m(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$, we have seen in many examples, that m-th Chern classes are often a product of first Chern classes. This is not expected: If $\mathbb{V}$ is a vector bundle with a projectively flat connection on a projective variety X with trivial fundamental group, then $c_m(\mathbb{V}) = \frac{1}{R^m}\binom{R}{m} c_1(\mathbb{V})^m$, where $R = \mathrm{rk}(\mathbb{V})$. The spaces $\overline{\mathrm{M}}_{0,n}$ are simply connected [BP00], and vector bundles of conformal blocks carry a projectively flat connection on the interior $\mathrm{M}_{0,n}$ [TUY89]. The connection for a vector bundle of conformal blocks on $\mathrm{M}_{0,n}$ does not extend to a projectively flat connection on the boundary.

Date: July 5, 2016.

2. Background and notation

The facts we use are given, primarily in the notation of [Fak12]. Original sources for the construction are [Tsu93, TUY89]. The stack $\overline{\mathcal{M}}_{0,n}$ is represented by the fine moduli space $\overline{\mathrm{M}}_{0,n}$, and we work on this space throughout.

2.1. **Basic ingredients.** A vector bundle of conformal blocks $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ is determined by a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of dominant weights for $\mathfrak{g}$ at level ℓ. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. A dominant integral weight $\lambda_i \in \mathfrak{h}^*$ is at level ℓ as long as $(\theta, \lambda_i) \leq \ell$, where $\theta \in \mathfrak{h}^*$ is the highest root of $\mathfrak{g}$, and $(\ ,\ )$ is the Killing form, normalized so that $(\theta, \theta) = 2$. We denote the set dominant integral weights of $\mathfrak{g}$ of level ℓ by $P_\ell(\mathfrak{g})$. For each weight λ_i, there is a unique and irreducible finite dimensional $\mathfrak{g}$-module V_{λ_i}.

2.2. **Vector spaces of conformal blocks.** Let $\mathfrak{g}$ be a simple Lie algebra, and let $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$, be the corresponding *affine Kac-Moody Lie algebra*. Here, c belongs to the center of $\hat{\mathfrak{g}}$, and the Lie bracket of $\hat{\mathfrak{g}}$ is given by the following rule:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)\,\mathrm{Res}_{t=0}\, g\frac{df}{dt}c,$$

where X, Y are elements of the Lie algebra $\mathfrak{g}$ and f, g are in $\mathbb{C}((t))$. For each $\lambda \in P_\ell(\mathfrak{g})$, there exists and unique irreducible, highest weight, integrable, $\hat{\mathfrak{g}}$-module H_λ with the following properties:

- H_λ is infinite dimensional;
- c acts on H_λ by a scalar ℓ; and
- $V_\lambda \hookrightarrow \mathrm{H}_\lambda$.

Let C be a connected projective curve over $\mathbb{C}$, and let $U \subset \mathrm{C}$ an open set. By $\mathfrak{g}(U)$ we mean the Lie algebra $\mathfrak{g} \otimes \mathcal{O}_\mathrm{C}(U)$. Let $p_1, \ldots, p_n \in \mathrm{C}$ be n smooth points and let $\lambda_1, \ldots, \lambda_n \in \mathcal{P}_\ell(\mathfrak{g})$. Choose a local coordinate ξ_i at each point p_i, and denote by f_{p_i} the Laurent expansion of any element $f \in \mathcal{O}_\mathrm{C}(\mathrm{C} \setminus \{p_1, \ldots, p_n\})$. Then for each i, there is a ring homomorphism

$$\mathcal{O}_\mathrm{C}(\mathrm{C} \setminus \{p_1, \ldots, p_n\}) \to \mathbb{C}((\xi_i)), \quad f \mapsto f_{p_i},$$

Since the sum of the residues of a meromorphic one-form is zero, it follows that there is an embedding of Lie algebras

$$\mathfrak{g}(\mathrm{C} \setminus \{p_1, \ldots, p_n\}) \hookrightarrow \hat{\mathfrak{g}}_n = \bigoplus_{i=1}^{n} \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C}c, \quad \mathrm{X} \otimes f \mapsto (\mathrm{X} \otimes f_{p_1}, \ldots, \mathrm{X} \otimes f_{p_n}, 0).$$

Set $\mathrm{H}_{\vec{\lambda}} = \mathrm{H}_{\lambda_1} \otimes \cdots \otimes \mathrm{H}_{\lambda_n}$. There is a natural $\hat{\mathfrak{g}}_n$-module structure on $\mathrm{H}_{\vec{\lambda}}$; and so it inherits a $\mathfrak{g}(\mathrm{C} \setminus \{p_1, \ldots, p_n\})$-module structure.

Definition 2.1. *Let $\mathfrak{a}$ be a Lie algebra. Let* V *and* W *be two $\mathfrak{a}$-modules. The space of coinvariants $[\mathrm{V} \otimes \mathrm{W}]_\mathfrak{a}$ is equal to the quotient of $\mathrm{V} \otimes \mathrm{W}$ by the subspace spanned by the elements of the form $\mathrm{X}v \otimes w + v \otimes \mathrm{X}w$, where $\mathrm{X} \in \mathfrak{a}$, $v \in \mathrm{V}$, and $w \in \mathrm{W}$.*

Definition 2.2. *With the notation above, set*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C;\vec{p})} = [\mathrm{H}_{\vec{\lambda}}]_{\mathfrak{g}(\mathrm{C} \setminus \{p_1,\ldots,p_n\})}, \quad \textit{and} \quad \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^{\dagger}_{(C;\vec{p})} = \mathrm{Hom}_{\mathfrak{g}(\mathrm{C} \setminus \{p_1,\ldots,p_n\})}(\mathrm{H}_{\vec{\lambda}}, \mathbb{C}),$$

where $\mathbb{C}$ is considered a trivial $\mathfrak{g}(\mathrm{C} \setminus \{p_1, \ldots, p_n\})$-module. That is, $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C;\vec{p})}$ is the space of coinvariants of the $\mathfrak{g}(\mathrm{C} \setminus \{p_1, \ldots, p_n\})$-module $\mathrm{H}_{\vec{\lambda}}$, which is the largest quotient of $\mathrm{H}_{\vec{\lambda}}$ on which $\mathfrak{g}(\mathrm{C} \setminus \{p_1, \ldots, p_n\})$ acts *trivially. This is equal to the quotient of* $\mathrm{H}_{\lambda_1}\otimes\cdots\otimes\mathrm{H}_{\lambda_n}$ *by the subspace spanned by elements of the form*

$$\sum_{i=1}^{n} v_1\otimes\cdots\otimes v_{i-1}\otimes(\mathrm{X}\otimes f_{p_i})\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n,$$

where $\mathrm{X}\otimes f\in\mathfrak{g}(C\setminus\{p_1,\dots,p_n\})$*, and* $v_i\in\mathrm{H}_{\lambda_i}$ *for all* $i\in\{1,\dots,n\}$*.*

2.3. **Brief sketch of construction of the sheaf of conformal blocks.** One can carry the construction done above out in families. To begin with, we let $S=\mathrm{Spec}(\mathrm{A})$ for some k-algebra A, and $\pi: C\to S$ a proper flat family of curves whose fibers have at worst ordinary double point singularities. For $1\le i\le n$, let $s_i: S\longrightarrow C$ be sections of π whose images are disjoint and contained in the smooth locus of π. In particular, over a point $s\in S$, we have an n-pointed curve $(C_s; s_1(s),\dots,s_n(s))$. We note that for each i there are isomorphisms $\widehat{\mathcal{O}}_{C,\eta_i}\xrightarrow{\sim}\mathrm{A}[[\xi_i]]$. where η_i is the generic point of the image $s_i(S)$. The inclusion $\mathcal{O}_{C,\eta_i}\hookrightarrow\widehat{\mathcal{O}}_{C,\eta_i}$, induces, for each $i\in\{1,\dots,n\}$, inclusions $\mathrm{FracField}(\mathcal{O}_{C,\eta_i})\hookrightarrow\mathrm{FracField}(\widehat{\mathcal{O}}_{C,\eta_i})\cong\mathrm{A}((\xi_i))$. For $U=C\setminus\cup_{i=1}^{n}\eta_i$, we associate to $C\to S$ the Lie algebra $\mathfrak{g}(U)=\mathfrak{g}\otimes_k\mathcal{O}_C(U)$, with Lie bracket given by $[\mathrm{X}\otimes f,\mathrm{Y}\otimes g]=[\mathrm{X},\mathrm{Y}]_{\mathfrak{g}}\otimes fg$. Using the isomorphisms, there are maps $\mathcal{O}_C(U)\to\mathrm{A}((\xi_i))$ which in turn give rise to injective maps

$$\mathfrak{g}(U)\to\Big(\bigoplus_{i=1}^{n}\mathfrak{g}\otimes_k\mathrm{A}((\xi_i))\Big)\oplus\mathbb{C}c\cong\hat{\mathfrak{g}}_n\otimes_k\mathrm{A}\,.$$

One can show that $\mathfrak{g}(U)$ is a Lie sub-algebra of $\hat{\mathfrak{g}}_n\otimes_k\mathrm{A}$, and $\mathrm{H}_{\vec{\lambda}}\otimes_k\mathrm{A}$ is a representation of $\hat{\mathfrak{g}}_n\otimes_k\mathrm{A}$.

Definition 2.3. *With the notation above, set*

$$\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(\pi:C\to S;\{s_i:S\to C\}_{i=1}^{n})}=[\mathrm{H}_{\vec{\lambda}}\otimes\mathrm{A}]_{\mathfrak{g}(U)},$$

and

$$\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|^{\dagger}_{(\pi:C\to S;\{s_i:S\to C\}_{i=1}^{n})}=\mathrm{Hom}_{\mathfrak{g}(U)}(\mathrm{H}_{\vec{\lambda}}\otimes\mathrm{A},\mathrm{A}),$$

where A *is considered a trivial* $\mathfrak{g}(U)$*-module.*

To define $\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(\pi:C\to S;\{s_i:S\to C\}_{i=1}^{n})}$ for in case S is not affine, take an open affine covering and extend by the sheaf property. In this description, the open set $\mathcal{C}\setminus\cup_{i=1}^{n}\eta_i$ has been implicitly assumed to be affine. But this premise can be removed using a descent argument: See [Fak12, Prop 2.1], and the discussion following.

2.4. **Global generation in case** $g=0$**.** The bundles $\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)$ are globally generated on $\overline{\mathcal{M}}_{0,n}$: There is a surjection

$$\mathbb{A}(\mathfrak{g},\vec{\lambda})=\mathrm{A}_{\mathfrak{g},\vec{\lambda}}\times\overline{\mathrm{M}}_{0,\mathrm{n}}\twoheadrightarrow\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell),\quad\text{where }\ \mathrm{A}_{\mathfrak{g},\vec{\lambda}}=[\otimes_{i=1}^{n}\mathrm{V}_{\lambda_i}]_{\mathfrak{g}},$$

is the vector space of coinvariants, the largest quotient space of $\otimes_{i=1}^{n}\mathrm{V}_{\lambda_i}$ on which $\mathfrak{g}$ acts trivially. This gives rise to a morphism $f_{\mathbb{V}}$ from $\overline{\mathrm{M}}_{0,\mathrm{n}}$ to the Grassmannian of $\mathrm{rk}\,\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)$ quotients of the vector space $\mathrm{A}_{\mathfrak{g},\vec{\lambda}}$:

$$\overline{\mathrm{M}}_{0,\mathrm{n}}\xrightarrow{f_{\mathbb{V}}}\mathrm{G}=\mathrm{Grass}^{quo}(\mathrm{rk}\,\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell),\dim\mathrm{A}_{\mathfrak{g},\vec{\lambda}}),\quad x\mapsto\ (\mathrm{A}_{\mathfrak{g},\vec{\lambda}}\twoheadrightarrow\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_x).\tag{1}$$

Definition 2.4. *The first Chern class, called* ***the conformal blocks divisor***

$$\mathbb{D}(\mathfrak{g},\vec{\lambda},\ell)=c_1(\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)),$$

gives the composition of $f_{\mathbb{V}}$ *with the Plücker embedding* p *of the Grassmannian* G *into* $\mathbb{P}=\mathbb{P}^{N-1}$*, where* $N=\binom{\dim\mathrm{A}_{\mathfrak{g},\vec{\lambda}}}{\mathrm{rk}\,\mathbb{V}(\mathfrak{g},\vec{\lambda})}$.

2.4.1. *Global generation and Chern classes.* Global generation implies therefore that the higher Chern classes are nef-they nonnegatively intersect all effective cycles of complementary dimension on $\overline{\mathcal{M}}_{0,n}$. One way to see that global generation fails on $\overline{\mathcal{M}}_{g,n}$ for $g \geq 1$, is to exhibit effective curves on which the first Chern classes have negative degree. This was first done by Fakhruddin for $\mathfrak{sl}_2$ bundles using curves that arise as the images of maps from $\overline{\mathcal{M}}_{1,1}$.

3. Higher Chern classes from Chern character formula

In [MOP+14], a beautiful and simple formula for the total Chern character of a general vector bundle of conformal blocks on $\overline{\mathcal{M}}_{g,n}$ for all g, and all n is given. Here we use this to find an explicit formula for $c_k(\mathbb{V})$ on $\overline{\mathrm{M}}_{0,n}$.

Given a vector bundle of conformal blocks $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$, following [MOP+14], we set $w(\lambda) = \frac{(\lambda, \lambda+2\rho)}{2(\mathfrak{g}^*+\ell)}$, where $\mathfrak{g}^*$ is the dual Coxeter number, and ρ is half of the sum of the positive roots.

Theorem 3.1. *The m-th Chern class of $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$, is given by the formula:*

$$c_m(\mathbb{V}) = (-1)^m \sum_{\substack{(m_1,\dots,m_j)\in\mathbb{Z}^j_{\geq 0}\\ m_1+2m_2+\cdots+jm_j=m}} \prod_{k=1}^{j} \frac{(-\,\mathrm{p}_k(\mathbb{V}))^{m_k}}{m_k!k^{m_k}}, \tag{2}$$

$$\mathrm{p}_k(\mathbb{V}) = \sum_{\substack{\vec{k}=(k_1,\dots,k_{n+m})\in\mathbb{Z}^{n+m}_{\geq 0}, \sum_{i=1}^{n+m} k_i=k,\\ I=(I_1,\dots,I_m)}} \beta_I^{\vec{k}}\ \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}\delta_{I_1}^{k_{n+1}}\cdots\delta_{I_m}^{k_{n+m}}, \tag{3}$$

for $I = (I_1, \dots, I_m)$, sets $I_1, \dots, I_m \subsetneq \{1, \dots, n\}$ are nested or disjoint, $1 \in (I_1 \cup \cdots \cup I_m)^c$,

$$\beta_I^{\vec{k}} = (-1)^{\sum_{j=1}^m k_{n+j}} \binom{k}{k_1, \dots, k_{n+m}} \sum_{\substack{(\mu_1,\dots,\mu_m)\\ \in \mathcal{P}_\ell(\mathfrak{g})^m}} \prod_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} w(\lambda_i)^{k_i}\ w(\mu_j)^{k_{n+j}}\ \mathrm{rk}\,\mathbb{V}(\{\lambda(I_j)\cup\mu_j\})\,\mathrm{rk}\,\mathbb{V}(\{\lambda(I_j^C)\cup\mu_j^*\}), \tag{4}$$

where $\mu^* \in \mathcal{P}_\ell(\mathfrak{g})$ be the element with the property that $-\mu^*$ is the lowest weight of the weight space V_μ.

Proof. We will use that $[\mathrm{Ch}(\mathbb{V})]_k = \frac{1}{k!}p_k(\mathbb{V})$, where $p_k(\mathbb{V})$ are the k-th power sums of the Chern roots of the vector bundle, and Equation (2) which is proved in a number of places, eg. [Mea92]. As we show, the asserted formula then follows from [MOP+14]. In particular, we will show that one can obtain the explicit formula for the power sums that is given in Equation (3).

So for example, In particular, it follows that $[\mathrm{ch}(\mathbb{V})]_1 = \mathrm{c}_1(\mathbb{V})$. We can use this to recover Fakhruddin's formula for the first Chern class of a vector bundle $\mathbb{V}$ of conformal blocks on $\overline{\mathrm{M}}_{0,n}$ from the formula for the total Chern Character $\mathrm{Ch}(\mathbb{V})$ given in [MOP+14].

For $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$, on $\overline{\mathrm{M}}_{0,n}$, by [MOP+14, Theorem 1],

$$\mathrm{Ch}(\mathbb{V}) = \sum_{\Gamma,\vec{\mu}} (i_\Gamma)_* \Big(\prod_{\substack{\ell\in\Gamma\\ legs}} \mathrm{cont}(\ell) \prod_{\substack{v\in\Gamma\\ vertices}} \mathrm{cont}(v) \prod_{\substack{e\in\Gamma\\ edges}} \mathrm{cont}(e) \Big),$$

where one sums over all graphs Γ dual to stable n-pointed curves of genus zero, and vectors $\vec{mu}$ of attaching weights. For the degree one part $[\mathrm{Ch}(\mathbb{V})]_1$, we expand the power series, given Γ:

$$\prod_{\substack{\ell\in\Gamma\\ legs}} \mathrm{cont}(\ell) = \prod_{i=1}^{n} \exp(w(\lambda_i)\psi_i) = \exp(\sum_{i=1}^{n} w(\lambda_i)\psi_i).$$

The other power series that contribute come from the edges. To begin with, we write

$$f(t) = \frac{1-e^{wt}}{t} = \frac{1-\sum_{m=0}^{\infty}\frac{(wt)^m}{m!}}{t} = -\sum_{m=0}^{\infty}\frac{w^{m+1}t^m}{(m+1)!} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}w^{m+1}(-t)^m}{(m+1)!}.$$

Now given an edge $e \in \Gamma$, with corresponding attaching weight μ_e,

$$\mathrm{cont}(e) = f(\psi'_e + \psi_e'') = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}w(\mu_e)^{m+1}(-\psi'_e - \psi_e'')^m}{(m+1)!}$$

and using the key identity from [MOP+14, bottom page 15] the image i_{Γ_*} is

$$\sum_{m=0}^{\infty}\frac{(-1)^{m+1}w(\mu_e)^{m+1}(\delta_e)^{m+1}}{(m+1)!} = \sum_{m=0}^{\infty}\frac{(-w(\mu_e)\,\delta_e)^{m+1}}{(m+1)!} = \exp(-w(\mu_e)\,\delta_e).$$

So given a vector $\vec{\mu} = (\mu_{e_1}, \dots, \mu_{e_m})$ of attaching weights for Γ,

$$\prod_{\substack{e\in\Gamma\\ edges}} \mathrm{cont}(e) = \exp(\sum_{j=1}^{m}(-w(\mu_{e_j})\,\delta_{e_j})).$$

$$\text{(5)}\quad \mathrm{Ch}(\mathbb{V}) = \sum_{\substack{\Gamma \text{ with m edges}\\ \vec{\mu}=\{\mu_{e_1},\dots,\mu_{e_m}\}\in\mathcal{P}_\ell(\mathfrak{g})^m}} \prod_{\substack{v\in\Gamma\\ vertices}} \mathrm{cont}(v)\exp(\sum_{i=1}^{n} w(\lambda_i)\psi_i + \sum_{j=1}^{m}(-w(\mu_{e_j})\,\delta_{e_j}))$$

$$= \sum_{\substack{\Gamma \text{ with m edges}\\ \vec{\mu}=\{\mu_{e_1},\dots,\mu_{e_m}\}\in\mathcal{P}_\ell(\mathfrak{g})^m}} \prod_{\substack{v\in\Gamma\\ vertices}} \mathrm{cont}(v)\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\substack{k_1,\dots,k_{n+m}\geq 0\\ \sum_{i=1}^{n+m}k_i=k}}\binom{k}{k_1,\dots,k_{n+m}}\prod_{i=1}^{n}(w(\lambda_i)\,\psi_i)^{k_i}\prod_{j=n+1}^{n+m}(-w(\mu_{e_j})\,\delta_{e_j})^{k_j}.$$

For instance, in order to solve for the degree one part $[\mathrm{Ch}(\mathbb{V})]_1$, we are concerned with summing over trees having either zero or one edges. Doing so, one obtains:

$$\text{(6)}\quad [\mathrm{Ch}(\mathbb{V})]_1 = \mathrm{rk}(\mathbb{V})\sum_{i=1}^{n} w(\lambda_i)\psi_i - \sum_{\substack{\Gamma \text{ with 1 edge e}\\ \mu_e\in\mathcal{P}_\ell(\mathfrak{g})}} \mathrm{rk}(\mathbb{V}(\mu_e))\ \mathrm{rk}(\mathbb{V}(\mu_e^*))\ w(\mu_e)\,\delta_e$$

$$= \mathrm{rk}(\mathbb{V})\sum_{i=1}^{n} w(\lambda_i)\psi_i - \sum_{\substack{I\subset N,\\ \mu\in\mathcal{P}_\ell(\mathfrak{g})}} w(\mu)\ \mathrm{rk}(\mathbb{V}(\mu))\ \mathrm{rk}(\mathbb{V}(\mu))\,\delta_I,$$

since, if δ_e is the boundary divisor δ_I corresponding to the 1-edge stable graph Γ with attaching weight μ_e on the side with points labeled by I and μ_e^* on the side with points labeled by I^C, then

$$\mathbb{V}(\mu_e) = \mathbb{V}(\mathfrak{g}, \{\lambda_i | i \in I\} \cup \{\mu_e\}, \ell), \text{ and } \mathbb{V}(\mu_e^*) = \mathbb{V}(\mathfrak{g}, \{\lambda_i | i \in I^C\} \cup \{\mu_e^*\}, \ell).$$

Equation (6) is Fakhruddin's formula [Fak12, Muk16]. Similarly, for the degree k part $[\mathrm{Ch}(\mathbb{V})]_k$, we sum over trees having at most k edges. By doing this, one obtains

$$[\mathrm{Ch}(\mathbb{V})]_k = \sum_{\substack{\vec{k}=(k_1,\dots,k_{n+m})\in\mathbb{Z}_{\geq 0}^{n+m}, \sum_{i=1}^{n+m}k_i=k,\\ I=(I_1,\dots,I_m),\ \text{condition}\star}} \alpha_I^{\vec{k}}\ \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}\delta_{I_1}^{k_{n+1}}\cdots\delta_{I_m}^{k_{n+m}},$$

and such that for $k(n) = \sum_{i=1}^{n} k_i$, and $k(m) = \sum_{j=1}^{m} k_{n+j}$:

$$\alpha_I^{\vec{k}} = \frac{(-1)^{k(m)}}{k!}\binom{k}{k_1, \ldots, k_{n+m}} \sum_{\substack{\vec{\mu}=(\mu_1,\ldots,\mu_m)\\ \in \mathcal{P}_\ell(\mathfrak{g})^m}} \prod_{i=1}^{n} w(\lambda_i)^{k_i} \prod_{j=1}^{m} w(\mu_j)^{k_{n+j}} \operatorname{rk} \mathbb{V}(\{\lambda(I_j) \cup \mu_j\}) \operatorname{rk} \mathbb{V}(\{\lambda(I_j^C) \cup \mu_j^*\}).$$

Since $[\mathrm{Ch}(\mathbb{V})]_k = \frac{1}{k!}p_k(\mathbb{V})$, multiplying through by $k!$ to obtain $\beta_I^{\vec{k}}$, we are finished. □

Remark 3.2. *(1) Using a theorem of* [EV80], *N. Fakhruddin, in* [Fak12, Theorem 3.2] *has given a formula for the Newton classes $N_m(\mathbb{V})$ of a conformal blocks bundle $\mathbb{V}$ on $\overline{\mathrm{M}}_{0,n}$: Namely*

$$N_m(\mathbb{V}) = (-1)^m \sum_{\alpha_1+\cdots+\alpha_s=m} \binom{m}{\alpha} \mathrm{Tr}\left(\Gamma_1^{\alpha_1} \circ \cdots \circ \Gamma_s^{\alpha_s}\right)[B_1]^{\alpha_1} \ldots [B_s]^{\alpha_s},$$

where B_i, $i = 1,\ldots, s$ are boundary components, and Γ_i is the residue of the KZ connection along B_i. The matrices Γ_i's have also been computed in [Fak12] *explicitly.*

(2) In [BG16], *Belkale and Gibney show that The Chern classes of $\mathbb{V}(\mathfrak{sl}_r, \ell)$ on $\overline{\mathcal{M}}_g$ are quasi-polynomial in l for sufficiently large ℓ. It would be interesting to know if the same is true on $\overline{\mathrm{M}}_{0,n}$.*

4. Additive and critical level identities for higher Chern classes

In [BGM14a, Thm 1.3], it was proved that for bundles in type A, all first Chern classes vanish above what is called the critical level. Bundles in type A at the critical level have so-called partner bundles, and in [BGM14a, Prop 1.6] it was shown that first Chern classes of partner bundles are equal. We have shown the k-th Chern class of any bundle at the critical level can be written as a sum of products of Chern classes of its critical level partner. By [BGM14a, Prop 1.6], the sum of the ranks of the critical level partner bundles equal the rank of the bundle of coinvariants. By our analogue for higher Chern classes, if $\mathbb{V}$ is a critical level bundle, and its partner has rank one, $c_k(\mathbb{V})$ is a product of first Chern classes. In [BGM14b, Prop 4.1], it was proved that if certain ranks are satisfied, additive relations hold for first Chern classes. We have shown an additive identity for higher Chern classes holds.

Proposition 4.1. *For $\vec{\nu}_1 \in \mathcal{P}_{\ell_1}(\mathfrak{g})$, $\vec{\mu}_1 \in \mathcal{P}_{m_1}(\mathfrak{g})$, and $\vec{\nu}_1 + \vec{\mu}_1 \in \mathcal{P}_{\ell_1+m_1}(\mathfrak{g})$, suppose:*

$$\mathrm{rk}(\mathbb{V}(\mathfrak{g}, \vec{\nu}_1, \ell_1)) = 1, \text{ and } \mathrm{rk}(\mathbb{V}(\mathfrak{g}, \vec{\mu}_1, m_1)) = \mathrm{rk}(\mathbb{V}(\mathfrak{g}, \vec{\nu}_1 + \vec{\mu}_1, \ell_1 + m_1)) = \delta.$$

Then

$$\mathrm{c}_m(\mathbb{V}(\mathfrak{g}, \vec{\nu}_1 + \vec{\mu}_1, \ell_1 + m_1)) = \sum_{k=0}^{m} \binom{m+\delta-k}{k} c_1(\mathbb{V}(\mathfrak{g}, \vec{\nu}_1, \ell_1))^k \, c_{m-k}(\mathbb{V}(\mathfrak{g}, \vec{\mu}_1, m_1)).$$

Proof. This follows from the fact that Chern characters are multiplicative for tensor products, and by [BGM14b, Prop 2.1], which gives that $\mathbb{V}(\mathfrak{g}, \vec{\nu}_1 + \vec{\mu}_1, \ell_1 + m_1) \cong \mathbb{V}(\mathfrak{g}, \vec{\nu}_1, \ell_1) \otimes \mathbb{V}(\mathfrak{g}, \vec{\mu}_1, m_1)$. □

Example 4.2. *On $\overline{\mathrm{M}}_{0,7}$, the bundle $\mathbb{V}_1 = \mathbb{V}(\mathfrak{sl}_3, \{(2\omega_1)^2, (\omega_1)^5\}, 2)$ has rank 3, $\mathbb{V}_2 = \mathbb{V}(\mathfrak{sl}_3, \{(\omega_1)^6, 0\}, 1)$ has rank 1, and $\mathbb{V}_3 = \mathbb{V}(\mathfrak{sl}_3, \{(3\omega_1)^2, (2\omega_1)^4, \omega_1\}, 3)$ has rank 3. Using Proposition 4.1, $c_2(\mathbb{V}_3) = c_2(\mathbb{V}_1) + 3c_1(\mathbb{V}_1)c_1(\mathbb{V}_2) + c_1(\mathbb{V}_2)^2$. In fact, this simplifies further, as we'll see later in Example 4.5.*

Definition 4.3. *(*[BGM14a]*) Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple of* normalized *integral weights for $\mathfrak{sl}(r+1)$, assume that $r+1$ divides $\sum_{i=1}^{n} |\lambda_i|$, and define the critical level for the pair $(\mathfrak{sl}(r+1), \vec{\lambda})$ to be*

$$\mathrm{CL}(\mathfrak{sl}(r+1), \vec{\lambda}) = -1 + \frac{1}{r+1}\sum_{i=1}^{n} |\lambda_i|.$$

A vector bundle $\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)(\mathfrak{sl}(r+1),\ell)$ *is said to be a critical level bundle if* $\ell = \mathrm{CL}(\mathfrak{sl}(r+1),\vec{\lambda})$ *and* $\vec{\lambda} \in P_\ell(\mathfrak{sl}(r+1))^n$. *We say that* $\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)$ *is above the critical level if* $\ell > \mathrm{CL}(\mathfrak{sl}(r+1),\vec{\lambda})$, *and below the critical level if* $\ell < \mathrm{CL}(\mathfrak{sl}(r+1),\vec{\lambda})$.

By [BGM14a, Proposition 1.3], if $\ell > \mathrm{CL}(\mathfrak{sl}(r+1),\vec{\lambda})$, then $\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)$ is a constant bundle and so all of its Chern classes are trivial. On the other hand, if $\ell = \mathrm{CL}(\mathfrak{sl}(r+1),\vec{\lambda})$, then $\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)$ has a partner bundle $\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,r)$ where the weight λ_i^T is obtained by taking the transpose of the Young diagram associated to the weight λ_i. Moreover, by [BGM14a, Proposition 1.6 (b)]:

$$c_1(\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)) = c_1(\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,r)).$$

This was discovered by Fakhruddin for $r=1$ [Fak12].

The following is a critical level identity for higher Chern classes:

Proposition 4.4. *If* $\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)$ *and* $\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,r)$ *are critical level partner bundles, then*

(7) $c_k(\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)) =$

$$\sum_{\substack{1\le n_1,\cdots,n_j\le k \\ \sum_{i=1}^j i\cdot n_i = k}} (-1)^{k-(n_1+\cdots+n_j)} \binom{\sum_{i=1}^j n_i}{n_1,\cdots,n_j} c_1(\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,r))^{n_1}\cdots c_j(\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,r))^{n_j}.$$

Proof. In [BGM14a] it was shown that one has the short exact sequence

$$0 \longrightarrow \mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)^* \hookrightarrow \mathbb{A}^*_{\mathfrak{sl}(r+1),\vec{\lambda}} \twoheadrightarrow \mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,r) \longrightarrow 0.$$

The bundle $\mathbb{A}^*_{\mathfrak{sl}(r+1),\vec{\lambda}}$ is constant, and so it's Chern classes are all zero. Now the result follows from the fact that Chern polynomials are multiplicative in short exact sequences.

□

Example 4.5. *In Example 4.2 we saw by Proposition 4.1,* $c_2(\mathbb{V}_3) = c_2(\mathbb{V}_1) + 3c_1(\mathbb{V}_1)c_1(\mathbb{V}_2) + c_1(\mathbb{V}_2)^2$. *Since* $\mathbb{V}_1$ *is at the critical level, we can write this as* $c_2(\mathbb{V}_3) = c_1(\mathbb{V}_1)^2 + 3c_1(\mathbb{V}_1)c_1(\mathbb{V}_2) + c_1(\mathbb{V}_2)^2$.

Example 4.6. *For any critical level bundle* $\mathbb{V}(\mathfrak{sl}(2),\vec{\lambda},\ell)$,

$$c_k(\mathbb{V}(\mathfrak{sl}(2),\vec{\lambda},\ell)) = c_1(\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,1))^k = c_1(\mathbb{V}(\mathfrak{sl}(2),\vec{\lambda},\ell))^k.$$

It's CL partner is $\mathbb{V}(\mathfrak{sl}(\ell+1),\vec{\lambda}^T,1)$, *and any level one bundle in type* A *has rank one by* [Fak12].

Example 4.7. *In* [Kaz14], *Kazanova proved that all* S_n*-invariant bundles for* $\mathfrak{sl}(n)$ *on* $\overline{\mathrm{M}}_{0,n}$ *of rank one are of the form* $\mathbb{V}(\mathfrak{sl}(n),\lambda^n,\ell)$, *where* $\lambda = (\ell-m)\omega_i + m\omega_{i+1}$. *For the bundle to be at the critical level we must also have* $i=1$ *and* $m=1$. *The dual partner is of the form* $\mathbb{V}(\mathfrak{sl}(r+1),(\omega_1+\omega_r)^n,n-1)$, *and by* [BGM14a] *the partner will have rank equal to the rank of the bundle of coinvariants minus one (In particular, for these bundles, ranks are ???). One has:*

$$c_k(\mathbb{V}(\mathfrak{sl}(r+1),(\omega_1+\omega_r)^n,n-1)) = c_1(\mathbb{V}(\mathfrak{sl}(r+1),(\omega_1+\omega_r)^n,n-1))^k = c_1(\mathbb{V}(\mathfrak{sl}(n),((r-1)\omega_1+\omega_2)^n,r))^k.$$

She also proved that one can express such divisors as sums of first Chern classes of level one bundles:

$$c_1(\mathbb{V}(\mathfrak{sl}(n),((r-1)\omega_1+\omega_2)^n,r)) = (r-1)c_1(\mathbb{V}(\mathfrak{sl}(n),(\omega_1)^n,1)) + c_1(\mathbb{V}(\mathfrak{sl}(n),(\omega_2)^n,1)) = c_1(\mathbb{V}((\mathfrak{sl}(n),\omega_2^n,1)),$$

since $c_1(\mathbb{V}(\mathfrak{sl}(n),(\omega_1)^n,1))$ *is above the critical level, and hence is zero. So*

$$c_k(\mathbb{V}(\mathfrak{sl}(r+1),(\omega_1+\omega_r)^n,n-1)) = c_1(\mathbb{V}((\mathfrak{sl}(n),\omega_2^n,1))^k, \quad \textit{for all } r.$$

In particular, this gives an infinite family of S_n -invariant bundles of rank equal to codimension one in its coinvariants, whose higher Chern classes are all powers of first Chern classes of the same bundle of level one.

Example 4.8. *On $\overline{\mathrm{M}}_{0,n}$ we have that, for any critical level bundle $\mathbb{V}(\mathfrak{sl}(2), \{m\omega_1, \dots, m\omega_1\}, 2m-1)$, satisfying (fill in definition of weight content maximal for $\mathfrak{sl}_2$), then*

$$c_k(\mathbb{V}(\mathfrak{sl}(2), \{m\omega_1, \dots, m\omega_1\}, 2m-1)) = m^k c_1(\mathbb{V}(\mathfrak{sl}(2), \{\omega_1, \dots, \omega_1\}, 1))^k.$$

Indeed, $\mathbb{V}(\mathfrak{sl}(2), \{m\omega_1, \dots, m\omega_1\}, 2m-1)$ is a critical level bundle whose critical level partner bundle $\mathbb{V}(\mathfrak{sl}(2m), \{\omega_m, \dots, \omega_m\}, r)$ has rank one by [Hob15]*. We get*

$$\begin{aligned}
(8)\quad & c_k(\mathbb{V}(\mathfrak{sl}(2), \{m\omega_1, \dots, m\omega_1\}, 2m-1)) \\
&= c_1(\mathbb{V}(\mathfrak{sl}(2), \{m\omega_1, \dots, m\omega_1\}, 2m-1))^k = c_1(\mathbb{V}(\mathfrak{sl}(2m), \{\omega_m, \dots, \omega_m\}, 1))^k \\
&= m^k c_1(\mathbb{V}(\mathfrak{sl}(2), \{\omega_1, \dots, \omega_1\}, 1))^k.
\end{aligned}$$

This example can be generalized to critical level type A bundles for which the weight content is maximal (cf [Hob15] *for a definition), showing their k-th Chern classes are also k-th power of their first Chern classes.*

5. Criteria for extremality

5.1. Extremality from the critical level.

Definition 5.1. *Let $\cup_{1 \le j \le k+3} J_j = N$ be a partion of $N = \{1, \dots, n\}$ into $k+3$ nonempty sets J_j. Let $Z_{\vec{J}}$ be the image in $\overline{\mathrm{M}}_{0,n}$ of the clutching map $\overline{\mathrm{M}}_{0,k+3} \twoheadrightarrow Z_{\vec{J}} \hookrightarrow \overline{\mathrm{M}}_{0,n}$ defined by sending a point $X = (C, p_1, \dots, p_{k+3}) \in \overline{\mathrm{M}}_{0,k+3}$ to a point in $\overline{\mathrm{M}}_{0,n}$, given by attaching $k+3$-points $(\mathbb{P}^1, J_j \cup \{q_j\})$ to X by identifying p_j with q_j.*

Proposition 5.2. *Suppose $r \ge 1$ and $\ell \ge 1$ and let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ be an n-tuple in $P_\ell(\mathfrak{sl}(r+1))$. Let $J_1, \dots, J_{k+3}$ be any partition of N into $k+3$ non-empty sets. Without loss of generality assume that $\lambda(J_i) = \sum_{a \in J_i} |\lambda_i|$ are ordered, i.e. $\lambda(J_1) \le \cdots \le \lambda(J_{k+3})$. If $\sum_{i=1}^{k+2} \lambda(J_i) \le \ell + r$, then $c_k(\mathbb{V}(\mathfrak{sl}(r+1), \vec{\lambda}, \ell)$ (possibly trivial) contracts the k-dimensional F-cycle $Z_{\vec{J}}$. In particular, $c_k(\mathbb{V}(\mathfrak{sl}(r+1), \vec{\lambda}, \ell))$ is extremal in* $\mathrm{Nef}^k(\overline{\mathrm{M}}_{0,n})$.

To prove Proposition 5.2 we use the factorization theorem in [TUY89] and vanishing above the critical level. We now recall the factorization theorem from [TUY89] for the convenience of the reader.

Theorem 5.3 (Factorization). [TUY89] *Let $(C_0; p_1, \dots, p_n)$ be a stable n-pointed curve of genus 0 where C_0 has a node x_0. Let $\nu : C_1 \sqcup C_2 \to C_0$ the normalization of C_0 at x_0 and $\nu^{-1}(x_0) = \{x_1, x_2\}$, with $x_i \in C_i$, then the fiber $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})}$ is isomorphic to*

$$\bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \lambda(C_1) \cup \{\mu\}, \ell)|_{(C_1;\{p_i \in C_1\} \cup \{x_1\})} \otimes \mathbb{V}(\mathfrak{g}, \lambda(C_2) \cup \{\mu^*\}, \ell)|_{(C_2;\{p_i \in C_2\} \cup \{x_2\})},$$

where $\lambda(C_i) = \{\lambda_j | p_j \in C_i\}$.

Proof. (of Proposition 5.2) The k-dimensional F-cycle $Z_{\vec{J}}$ lies in a boundary component of $\overline{\mathrm{M}}_{0,n}$ that is isomorphic to the k-cycle in $\overline{\mathrm{M}}_{0,|J_1 \cup \cdots \cup J_{k+2}|+1} \times \{pt\}$ under the attaching map described in Definition 5.1. To show $c_k(\mathbb{V}(\mathfrak{sl}(r+1), \vec{\lambda}, \ell))$ contracts $Z_{\vec{J}}$ we can use Factorization to examine $\mathbb{V}(\mathfrak{sl}(r+1), \vec{\lambda}, \ell)$ at points that lie in the boundary component. By factorization, at points in $\overline{\mathrm{M}}_{0,|J_1\cup\cdots\cup J_{k+2}|+1}\times\overline{\mathrm{M}}_{0,|J_{k+3}|+1}$, the conformal block bundle decomposes as a direct sum, where each factor is of the form

$$\mathbb{V}(\mathfrak{sl}(r+1),\{\lambda_i\}_{i\in J}\cup\mu,\ell)\otimes\mathbb{V}(\mathfrak{sl}(r+1),\{\lambda_i\}_{i\in J_{k+3}}\cup\mu^*,\ell),$$

such that $\mu\in P_\ell(\mathfrak{sl}(r+1))$ and $J=J_1\cup\cdots\cup J_{k+2}$. We note that $|\mu|$ is bounded by $r\cdot\ell$. Hence $\sum_{i\in J}|\lambda_i|+|\mu|\leq\ell+r+r\cdot\ell<(\ell+1)(r+1)$. This implies that ℓ is above the critical level for $\mathbb{V}(\mathfrak{sl}(r+1),\{\lambda_i\}_{i\in J}\cup\mu,\ell)$. In [BGM14a], it was shown that in case ℓ is above the critical level, then the bundle itself is trivial. In particular, all Chern classes will be trivial. It follows that the k-th Chern class of $\mathbb{V}(\mathfrak{sl}(r+1),\vec{\lambda},\ell)$ pulled back to $\overline{\mathrm{M}}_{0,|J_1\cup\cdots\cup J_{k+2}|+1}\times\mathrm{pt}$ is trivial.

□

Example 5.4. *On $\overline{\mathrm{M}}_{0,2m}$, the bundle $\mathbb{V}(\mathfrak{sl}(2),\omega_1^{2m},m-1)$ has critical level partner $\mathbb{V}(\mathfrak{sl}(m),\omega_1^{2m},1)$, a bundle of rank one. By Section 4,*

$$c_k(\mathbb{V}(\mathfrak{sl}(2),\omega_1^{2m},m-1)=c_1(\mathbb{V}(\mathfrak{sl}(2),\omega_1^{2m},m-1)^k.$$

Let $J_i=\{i\}$ for $1\leq i\leq k+2$ and $J_{k+3}=\{1,\ldots,2m\}\setminus\cup_{i=1}^{k+2}J_i$. The sets $\vec{J}=(J_1,\ldots,J_{k+3})$ define a k-dimensional F-cycle under the attaching map $\overline{\mathrm{M}}_{0,k+3}\to\overline{\mathrm{M}}_{0,2m}$.

By Prop 5.2, to show the pull back of $\mathbb{V}(\mathfrak{sl}(2),\omega_1^{2m},m-1)$ under the ataching map is trivial, it is enough to guarantee the critical level of the bundle $\mathbb{V}(\mathfrak{sl}(2),\omega_1^{k+2},\mu;m-1)$ on $\overline{\mathrm{M}}_{0,k+3}$ is less than $m-1$. This is true for example when $k+2\leq m$. More generally, if $k+2\leq m$, then $c_k(\mathbb{V}(\mathfrak{sl}(2),\omega_1^{2m},m-1)$ are all extremal in $\mathrm{Nef}^k(\overline{\mathrm{M}}_{0,2m})$.

5.2. **Extremality from the theta level.** The theta level (Def 5.5), comes from the interpretation of a vector space of conformal blocks as an explicit quotient [Bea96, Proposition 4.1] see also [FSV90]), and holds in all types. It was also shown in [BGM14a], that conformal blocks divisors above the theta level are trivial.

Definition 5.5. [BGM14a] *Given a pair $(\mathfrak{g},\vec{\lambda})$, one refers to*

$$\theta(\mathfrak{g},\vec{\lambda})=-1+\frac{1}{2}\sum_{i=1}^{n}\lambda_i(H_\theta)\in\frac{1}{2}\mathbb{Z}$$

as the theta level. Here H_θ is the co-root corresponding to the highest root θ.

Proposition 5.6. [BGM14a] *Let $\vec{\lambda}\in\mathcal{P}_\ell(\mathfrak{g})^n$. Let $J_1,\ldots,J_{k+3}$ be any partition of N into $k+3$ non-empty sets. Without loss of generality assume that $\lambda(J_i)=\sum_{a\in J_i}|\lambda_i|$ are ordered, i.e. $\lambda(J_1)\leq\cdots\leq\lambda(J_{k+3})$. If $\sum_{i=1}^{k+2}\lambda(J_i)\leq\ell+1$, then $c_k(\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell))$ contracts the k-dimensional F-cycle $Z_{\vec{J}}$, and in particular, $c_k(\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell))$ is extremal in $\mathrm{N}^k(\overline{\mathrm{M}}_{0,n})$.*

The proof of Proposition 5.6 is analogous to that of Proposition 5.2.

6. Full dimensional subcones of Pliant cones

6.1. Fakhruddin's Basis and the Pliant Cone for $\overline{\mathrm{M}}_{0,n}$.

Claim 6.1. *There is a spanning set for $\mathrm{A}^m(\overline{\mathrm{M}}_{0,n})$, given by a basis of first Chern classes of vector bundles of conformal blocks. In particular, all classes lie in the pliant cone, and are extremal in $\mathrm{Nef}^m(\overline{\mathrm{M}}_{0,n})$.*

Proof. By [Kee92], $A^1(\overline{\mathrm{M}}_{0,n})$ generates $A^m(\overline{\mathrm{M}}_{0,n})$, all m [Kee92].

There is at least one basis we may use for the Picard group of $\overline{\mathrm{M}}_{0,n}$ (see Section 6.2 for more choices in the S_n invariant case). Namely, the bundles $\mathcal{B}$ that generate Fakhruddin's basis for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})$, are

$$\mathcal{B} = \{\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}, 1) : \mathrm{rk}(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}, 1) \neq 0\}.$$

In $\mathcal{B}$ bundles are determined by n-tuples of weights of the form $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \{0, \omega_1\}$, $0 \neq \sum_i |\lambda_i|$ is divisible by 2 and such that at least four weights λ_i are different than zero. Moreover, all have level one, and so also have rank one. We note that if n is odd, then all elements of $\mathcal{B}$ are pulled back from $\overline{\mathrm{M}}_{0,n-1}$ and if n is even, then $\mathbb{V}(\mathfrak{sl}(2), \{\omega_1^n\}, 1)$ is the unique element of $\mathcal{B}$ that is not pulled back from $\overline{\mathrm{M}}_{0,n-1}$.

The divisors generating the spanning set for $\mathrm{A}^m(\overline{\mathrm{M}}_{0,n})$ are all extremal: In case n is odd, then they are all pulled back from $\overline{\mathrm{M}}_{0,n-1}$ and intersect the fibral curve in the projection map in degree zero. If n is odd, then except for $E = c_1(\mathbb{V}(\mathfrak{sl}(2), \omega_1^n, 1))$, they are all pulled back from $\overline{\mathrm{M}}_{0,n-1}$ and so are extremal. In fact, by [AGS10], E intersects a number of F-curves in degree zero, and are therefore extremal. ◻

Example 6.2. *For instance:*

(1) *For $n \geq 7$, odd, then for $N = \{1, \ldots, n\}$, we have $A^2(\overline{\mathrm{M}}_{0,N})$ is spanned by classes of the form*
- $\pi_i^*(\beta_i) \cdot \pi_j^*(\beta_j)$, *where for $k \in \{i, j\}$, $\pi_k : \overline{\mathrm{M}}_{0,N} \to \overline{\mathrm{M}}_{0,N\setminus\{k\}}$, $\beta_k \in A^1(\overline{\mathrm{M}}_{0,N\setminus\{k\}})$;*
- $\pi_i^*(\alpha)$, *where $\pi_i : \overline{\mathrm{M}}_{0,N} \to \overline{\mathrm{M}}_{0,N\setminus\{i\}}$, $\alpha \in A^2(\overline{\mathrm{M}}_{0,N\setminus\{i\}})$.*

Using Fakhruddin's basis of first Chern classes of $\mathfrak{sl}_2$ divisors, this translates as:
- $c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}(i), 1)) \cdot c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\mu}(j), 1))$;
- $c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}(i), 1)) \cdot c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\mu}(i), 1))$,

where $\vec{\lambda}(i) = (\lambda_1, \ldots, \lambda_7)$ with $\lambda_i = 0$, and $\vec{\mu}(j) = (\mu_1, \ldots, \mu_7)$, where $\mu_j = 0$.

(2) *For $n \geq 6$, even, then there is a divisor $E \in A^1(\overline{\mathrm{M}}_{0,N})$, such that $A^2(\overline{\mathrm{M}}_{0,N})$ is spanned by*
- E^2, $E \cdot \pi_j^*(\beta_j)$, $\pi_i^*(\beta_i) \cdot \pi_j^*(\beta_j)$, *where $k \in \{i, j\}$, $\pi_k : \overline{\mathrm{M}}_{0,N} \to \overline{\mathrm{M}}_{0,N\setminus\{k\}}$, $\beta_k \in A^1(\overline{\mathrm{M}}_{0,N\setminus\{k\}})$;*
- $\pi_i^*(\alpha)$, *where $\pi_i : \overline{\mathrm{M}}_{0,N} \to \overline{\mathrm{M}}_{0,N\setminus\{i\}}$, $\alpha \in A^2(\overline{\mathrm{M}}_{0,N\setminus\{i\}})$.*

Using Fakhruddin's basis, this translates as $E = c_1(\mathbb{V}(\mathfrak{sl}(2), \omega_1^n, 1))$, and we have

(a) $c_1(\mathbb{V}(\mathfrak{sl}(2), \omega_1^n, 1))^2$, $c_1(\mathbb{V}(\mathfrak{sl}(2), \omega_1^n, 1)) \cdot c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}(i), 1))$, *and* $c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}(i), 1)) \cdot c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\mu}(j), 1))$;

(b) $c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}(i), 1)) \cdot c_1(\mathbb{V}(\mathfrak{sl}(2), \vec{\mu}(i), 1))$.

where $\vec{\lambda}(i) = (\lambda_1, \ldots, \lambda_8)$ with $\lambda_i = 0$, and $\vec{\mu}(j) = (\mu_1, \ldots, \mu_8)$, where $\mu_j = 0$.

Remark 6.3. *Swinarski showed that Fakhruddin's basis does not cover the whole nef cone of $\overline{\mathrm{M}}_{0,6}$, and so it isn't likely that these could be used to show that the cones spanned by conformal blocks classes and nef cones are the same for $k > 1$. However, there are other natural basis given by conformal blocks divisors, such as the set of bundles studied in* [AGSS11]*, which is known to form a basis for* $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$ *for $n \leq 2000$, and that studied in* [AGS10] *which is known to form a basis for* $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$ *for all n. These are (each) known to span the whole S_n-invariant cone of nef divisors, for $n \leq 8$. These are discussed in Section 6.2.*

To give some idea for how many more classes one gets than strictly necessary (reflecting identities), we computed the dimension of the vector space of cycles $A^k(\overline{\mathrm{M}}_{0,n})$.

Claim 6.4. *For $0 \le k \le \lfloor \frac{n-3}{2} \rfloor$, the dimension of the vector space of cycles $A^k(\overline{\mathrm{M}}_{0,n})$ and $A^{n-3-k}(\overline{\mathrm{M}}_{0,n})$ is*

$$1 + \sum_{j=1}^{k-1} j\binom{n-1}{j} + \ k \sum_{j=k}^{n-3-k} \binom{n-1}{j} + \sum_{j=n-2-k}^{n-4} (n-3-j)\binom{n-1}{j}.$$

Example 6.5. *(1) If $k=0$, the second and third sums are zero (or empty), so the number is 1;*

(2) If $k=1$, we get the familiar expression for the Picard number:

$$\sum_{i=0}^{n-4}\binom{n-1}{i} = \sum_{i=0}^{n-1}\binom{n-1}{i} - \binom{n-1}{n-3} - \binom{n-1}{n-2} - \binom{n-1}{n-1} = 2^{n-1} - \binom{n}{2} - 1.$$

(3) If $k=2$, the expression looks like:

$$1 + \binom{n-1}{1} + \ 2\sum_{j=2}^{n-5}\binom{n-1}{j} + \binom{n-1}{3}.$$

Note that using Pascale's identity, we can write this in the following way:

- $1 + \binom{n}{2} + \binom{n}{3} + 2\ \sum_{i=2}^{m}\binom{n}{2i}$, *if $n-5 = 2m$; and*
- $1 + \binom{n}{2} + \binom{n}{3} + 2\ \sum_{i=2}^{m}\binom{n}{2i} + 2\binom{n-1}{2m+1}$, *if $n-5 = 2m+1$.*

For example:

- $\dim(A^2(\overline{\mathrm{M}}_{0,6})) = 1 + \binom{6}{2} = 16$;
- $\dim(A^2(\overline{\mathrm{M}}_{0,7})) = 1 + \ \binom{7}{2} + \ \binom{7}{3} = 57$,
- $\dim(A^2(\overline{\mathrm{M}}_{0,8})) = 1 + \ \binom{8}{2} + \ \binom{8}{3} + 2\ \binom{7}{3} = 155$; *and*
- $\dim(A^2(\overline{\mathrm{M}}_{0,9})) = 1 + \ \binom{9}{2} + \ \binom{9}{3} + 2\ \binom{9}{4} = 373$.

Proof. (of Claim 6.4) This follows from Kapranov's construction of $\overline{\mathrm{M}}_{0,n}$ as a series of blowups of $\mathbb{P}^{n-3}$ along linear subspaces spanned by linear combinations of subsets of $(n-1)$-points in general linear position. One can then use the blowup formula for Poincare Polynomials. Namely, set $X_1 = Bl_{Z_0}(\mathbb{P}^{n-3})$, and $X_{i+1} = Bl_{Z_i}(X_i)$, where Z_i is the set of linear subspaces of dimension i spanned by all subsets of $\binom{n-1}{i+1}$ points. then $h_{X_{i+1}} = h_{X_i} + \binom{n-1}{i+1}(1+t^1+t^2+\cdots+t^i)(t^1+t^2+\cdots+t^{n-4-i})$. Adding everything together gives the assertion. □

6.2. **The S_n-invariant case.** *One may use a basis for $\mathrm{Pic}(\mathrm{M}_{0,n})^{\mathrm{S}_n}$ to generate a full-dimensional subcone of the S_n-invariant Pliant cone $\mathrm{Pl}(\mathrm{M}_{0,n})^{\mathrm{S}_n}$. Two such basis are given by the following sets: If $n = 2(g+1)$ is even, then by* [AGS10]

$$\mathcal{B}_1 = \{\mathbb{V}(\mathfrak{sl}(2), \omega_1^n, \ell) : 1 \le \ell \le g\},$$

and if $n = 2(g+1)+1$ is odd,

$$\mathcal{B}_1 = \{\mathbb{V}(\mathfrak{sl}(2), \{\omega_1^{n-1}, 0\}, \ell) : 1 \le \ell \le g\}$$

is a basis for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n}^{\mathrm{S}_n})$. And at least for $n \le 2,000$, by [AGSS11] *and a computer check,*

$$\mathcal{B}_2 = \{\mathbb{V}(\mathfrak{sl}(n), \{\omega_i^n\}, 1) : 2 \le i \le \lfloor\frac{n}{2}\rfloor\}.$$

is a basis for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$. The second basis $\mathcal{B}_2$ may be more interesting, as was shown in [AGSS11] *all of its elements span extremal rays of the nef cone $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$. All elements of $\mathcal{B}_2$ are of level one, and hence rank one, and so the only nontrivial classes will be products of first Chern classes. In contrast, only the first Chern classes of bundles with $\ell \in \{1,2,g-1,g\}$ span extremal rays of the nef cone. In particular, starting at $n = 12$, the $\mathcal{B}_2$ gives more extremal rays of the nef cone than does $\mathcal{B}_1$. Elements of $\mathcal{B}_1$ generally have rank* *greater than one, so products of first Chern classes will sometimes be equivalent to higher Chern classes: For instance, by the Critical Level Identity,*

$$c_1(\mathbb{V}(\mathfrak{sl}(2), \{\omega_1^{2(g+1)}\}, g))^k = c_k(\mathbb{V}(\mathfrak{sl}(2), \{\omega_1^{2(g+1)}\}, g)).$$

We give a few explicit examples below.

n	$[B_2, B_3]$	$\mathbb{V}$	*rank*	$\mathbb{V}$	*rank*
6	$[2,1]$	$\alpha_1 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^6\}, 1)$	*1*	$\alpha_1 = \mathbb{V}(\mathfrak{sl}_6, \{\omega_3\}, 1)$	*1*
6	$[1,3]$	$\alpha_2 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^6\}, 2)$	*2*	$\alpha_2 = \mathbb{V}(\mathfrak{sl}_6, \{\omega_2^6\}, 1)$	*1*
7	$[1,1]$	$\beta_1 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^6, 0\}, 1)$	*1*	$\beta_1 = \mathbb{V}(\mathfrak{sl}_7, \{\omega_3^7\}, 1)$	*1*
7	$[1,3]$	$\beta_2 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^6, 0\}, 2)$	*4*	$\beta_2 = \mathbb{V}(\mathfrak{sl}_7, \{\omega_2^7\}, 1)$	*1*
8	$[3,2,4]$	$\gamma_1 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^8\}, 1)$	*1*	$\gamma_1 = \mathbb{V}(\mathfrak{sl}_8, \{\omega_4^8\}, 1)$	*1*
8	$[6,11,8]$	$\gamma_2 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^8\}, 2)$	*8*	$\gamma_2 = \mathbb{V}(\mathfrak{sl}_8, \{\omega_3^8\}, 1)$	*1*
8	$[1,3,6]$	$\gamma_3 = \mathbb{V}(\mathfrak{sl}_2, \{\omega_1^8\}, 3)$	*13*	$\gamma_3 = \mathbb{V}(\mathfrak{sl}_8, \{\omega_2^8\}, 1)$	*1*

A spanning set for $A^2(\overline{\mathrm{M}}_{0,7}^{\mathrm{S}_7})$ is given by products of extremal rays of the cone of S_7-invariant nef divisors:

$$(\pi_7^*\alpha_1)^2 = \beta_1^2, \quad (\pi_7^*\alpha_1)\cdot(\pi_7^*\alpha_2) = \beta_1\cdot\beta_2, \quad \textit{and} \quad (\pi_7^*\alpha_2)^2 = \beta_2^2.$$

Moreover: $c_1(\mathbb{V}(\mathfrak{sl}_2, \{\omega_1^6, 0\}, 2))^2 = c_2(\mathbb{V}(\mathfrak{sl}_2, \{\omega_1^6, 0\}, 2))$, by the Critical Level identity. A spanning set for $A^2(\overline{\mathrm{M}}_{0,8}^{\mathrm{S}_8})$ of elements in the Pliant cone $Pl(\overline{\mathrm{M}}_{0,8}^{\mathrm{S}_8})$ is given by γ_1^2, γ_2^2, γ_3^2, $\gamma_1\cdot\gamma_2$, $\gamma_1\cdot\gamma_3$, and $\gamma_2\cdot\gamma_3$. Again by the Critical Level identity, $c_1(\mathbb{V}(\mathfrak{sl}_2, \{\omega_1^8\}, 2))^2 = c_2(\mathbb{V}(\mathfrak{sl}_2, \{\omega_1^8\}, 2))$. However, unlike with the previous case, these classes $\gamma_i\cdot\gamma_j$ aren't products of pullbacks of extremal rays of the cone of S_7-invariant nef divisors.

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