# Vector bundles of conformal blocks and $\overline{\mathcal{M}}_{g,n}$

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**Abstract**

The moduli space of stable n-pointed curves of genus g has played an important role in the literature: as a means of learning about smooth curves and their degenerations, as a model for moduli spaces generally, and as a test variety for developing theories in algebraic geometry. Conformal blocks are invariants of a curve attached to a Lie group. In particular, vector spaces of conformal blocks for G at any stable curve C can be identified with global sections of an ample line bundle on the moduli stack of G-bundles on C. These vector spaces fit together to form vector bundles, and we can use these bundles as a tool to study the moduli space of curves

These are notes from my lectures at GAeL about aspects of vector bundles of conformal blocks and the moduli space of curves. During the first lecture I introduce the moduli space of curves and illustrate my own interest in these bundles on the moduli space through a specific problem. In the second lecture I give one definition of the bundles, using affine Lie algebras mentioning the important Factorization and Propagation of Vacua theorems. In lecture 3 I focus on geometric interpretations of the bundles, and in Lecture 4 about Chern classes of the bundles.

In these notes I've tried to include more detail than I had time for in my talks, although I'm sorry that I have left a lot out.

# Contents

# Lecture 1

# VBs of CBs and $\overline{\mathrm{M}}_{g,n}$

## 1.1 Introduction

In these four lectures I am going to be talking about vector bundles of conformal blocks on moduli spaces of curves. I'll mention a number of open questions here, and there are more problems I could talk about if there was more time. Please contact me and keep in touch if you become interested.

The goals of my lecture today are to:

1. Introduce the moduli space of curves $\overline{\mathrm{M}}_{g,n}$; and
2. Motivate how vector bundles of conformal blocks (VBs of CBs) on $\overline{\mathrm{M}}_{g,n}$ have the potential to help us better understand $\overline{\mathrm{M}}_{g,n}$.

In these notes I have tried to put in more detail and references than I gave in my lectures.

The moduli space of curves remains a fundamental object of study in algebraic geometry for a number of reasons. For instance it is a:

- useful tool for studying smooth curves and their degenerations;
- prototype for what one wants to achieve when constructing a moduli space;
- good test space, having accessible structure:
  - It has a stratification by the topological type of curve being parametrized;
  - $\mathrm{S}_n$ acts on $\overline{\mathrm{M}}_{g,n}$ by permuting the marked points, giving $\overline{\mathrm{M}}_{g,n}$ combinatorial structure similar to a homogeneous variety.

As I'll illustrate with an example about cones of nef divisors, this last feature often enables one to reduce problems for moduli of higher genus curves to $\overline{\mathrm{M}}_{0,n}$.

Conformal blocks are invariants of a curve $C$ attached to a Lie group G. In particular, vector spaces of conformal blocks for G at any stable curve $C$ can be identified with global sections of an ample line bundle on the moduli stack of G-bundles on $C$. For instance, if $G = \mathrm{SL}(r)$ and $C$ is a smooth curve, these are global sections of the theta divisor on the moduli space of (semistable) vector bundles on $C$ of rank $r$ with trivializable determinant.

Ultimately, both the subjects of moduli of curves and conformal blocks are aimed at understanding aspects of curves through the study of moduli on them. Because they play such a central role, it is natural to begin the lectures with a primer on the moduli space of curves. In these notes I will also say more about moduli spaces more generally. For further reading, see [Kol96, Chapter 1], [EH00, Chapter VI, page ], Kleiman's article on the Picard Scheme in [FGI+05], and [HM98].

## 1.2 Moduli spaces in general

### 1.2.1 Moduli problems

Moduli spaces are solutions to moduli problems. To describe a moduli problem, we will start with any reasonable class of objects $\mathcal{S}$ which we may wish to study; For example:

- all smooth and proper curves of a fixed genus $g$ defined over a field $k$;
- all curves, defined over $k$, of a fixed genus $g \geqslant 1$ with one non-separating node;
- given a smooth curve $C$, all vector bundles of rank $r$ on $C$, up to isomorphism.

As part of being a reasonable collection of objects, $\mathcal{S}$ should be closed under base extension. So for example, if objects X in $\mathcal{S}$ are defined over $\mathrm{Spec}(k)$, where $k$ is a field, and if $k \hookrightarrow \mathrm{k}$ is a field extension, then $\mathrm{X}_{\mathrm{k}} = \mathrm{X} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathrm{k})$ is also in $\mathcal{S}$.

**Definition 1.2.1.** *Given a reasonable collection of objects $\mathcal{S}$ as above, we define a (contravariant) moduli functor from the category $(\mathcal{S}ch_k)$ of schemes over k to the category $(\mathcal{S}ets)$ of sets*

$$\mathcal{F}_{\mathcal{S}} : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \mathcal{F}_{\mathcal{S}}(\mathrm{T}),$$

*where $\mathcal{F}_{\mathcal{S}}(\mathrm{T})$ is equal to the set of flat families of objects in $\mathcal{S}$ parametrized by* T *up to isomorphism over* T.

One then asks:

**Question 1.2.2.** *Is there a flat morphism of schemes:*

$$u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}},$$

*which is a fine moduli space for the moduli functor?*

If the answer to Question 1.2.2 is yes, then for every object $T \in \mathrm{Obj}(\mathcal{S}ch_k)$, pulling back, gives an equivalence of sets:

$$\mathcal{F}_{\mathcal{S}}(T) = \mathcal{M}or_{\mathcal{S}ch}(T, \mathrm{Mod}_{\mathcal{S}}).$$

For example, taking $T = \mathrm{Mod}_{\mathcal{S}}$, we obtain the universal family

$$u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}}$$

which corresponds to the identity element

$$\mathrm{id} \in \mathcal{M}or_{\mathcal{S}ch}(\mathrm{Mod}_{\mathcal{S}}, \mathrm{Mod}_{\mathcal{S}}).$$

And taking $T = \mathrm{Spec}(k)$, we see that the set of $k$-points of $\mathrm{Mod}_{\mathcal{S}}$ corresponds to the fibers of the family $u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}}$.

Another more formal way to say this is the following.

**Definition 1.2.3.** *The functor $\mathcal{F}_{\mathcal{S}}$ from Definition 1.2.1 is represented by the scheme* $\mathrm{Mod}_{\mathcal{S}}$ *if there is a natural isomorphism between $\mathcal{F}_{\mathcal{S}}$ and the functor of points $\mathcal{M}or_{\mathcal{S}ch}(\quad, \mathrm{Mod}_{\mathcal{S}})$. In this case we say* $\mathrm{Mod}_{\mathcal{S}}$ *is* ***a fine moduli space*** *for the functor $\mathcal{F}_{\mathcal{S}}$.*

### 1.2.2 The functor of points

**Definition 1.2.4.** *Let X be a scheme over a field k. The* ***functor of points*** *of a scheme X is the contravariant functor*

$$h_X : (\mathcal{S}ch_k) \to (\mathcal{S}ets),$$

*from the category $(\mathcal{S}ch_k)$ of schemes over k to the category $(\mathcal{S}ets)$ of sets which takes a scheme $Y \in \mathrm{Ob}(\mathcal{S}ch_k)$ to the set $h_X(Y) = \mathcal{M}or_{\mathcal{S}ch_k}(Y, X)$, and takes maps of schemes $f : Y \to Z$, to maps of sets:*

$$h_X(f) : h_X(Z) \to h_X(Y), \;\; [g : Z \to X] \mapsto [g \circ f : Y \to X].$$

**Definition 1.2.5.** *We say that a contravariant functor*

$$F : (\mathcal{S}ch_k) \to (\mathcal{S}ets),$$

***is representable*** *if it is of the form $h_X$ for some scheme X. By Yoneda's Lemma (below), if X exists, then it is unique, and we say that X represents the functor F.*

For a proof of **Yoneda's Lemma**, which we next state, see for example [EH00, pages 252-253]

**Lemma 1.2.6** (Yoneda)**.** *Let $C$ be a category and X, and let $X' \in \mathrm{Obj}(C)$.*

1. *If F is any contravariant functor from $C$ to the category of sets, the natural transformations from $\mathcal{M}or(,X)$ to F are in natural correspondence with the elements of $F(X)$;*
2. *If functors $\mathcal{M}or(,X)$ and $\mathcal{M}or(,X')$ are isomorphic, then $X \cong X'$.*

### 1.2.3 $M_g$: Not a fine moduli space

Consider, for $g = \dim H^1(C, O) \geqslant 2$:

$$\mathcal{M}_g : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{M}_g(T),$$

where $\mathcal{M}_g(T)$ is the set of proper flat maps $\pi : \mathcal{F} \to T$ such that every fiber $\mathcal{F}_t$ is a smooth projective curve of genus $g$ modulo isomorphism over T. This functor is not represented by a fine moduli space: every curve with nontrivial automorphisms creates issues.

**Example 1.2.7.** *We will consider a nontrivial family of hyperelliptic curves parametrized by $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$. To describe this family, let $X = Z(y^2 - f(x))$ be any smooth hyperelliptic curve of genus g with $\mathrm{Aut}(X) \cong C_2 =< \tau >$. The cyclic group $C_2$ acts on X and on $\mathbb{G}_m$:*

$$C_2 \times X \to X, \ (\tau, (x, y)) \mapsto (x, -y), \quad \textit{and} \quad C_2 \times \mathbb{G}_m \to \mathbb{G}_m, \ (\tau, z) \mapsto -z;$$

*and we can form the contracted product*

$$\mathcal{F} = \mathbb{G}_m \times_{C_2} X = (\mathbb{G}_m \times X)/\sim, \quad \textit{where} \quad (\tau \cdot \alpha, p) \sim (\alpha, \tau \cdot p).$$

*We'll set*

$$\pi : \mathcal{F} \to \mathbb{G}_m \ \ [(\alpha, p)] \mapsto \alpha^2,$$

*which is well defined since by this prescription* $(\tau \cdot \alpha, p) = (-\alpha, p) \mapsto \alpha^2$, *and* $(\alpha, \tau \cdot p) \mapsto \alpha^2$. *To see that fibers of* $\pi$ *are isomorphic to* X, *notice that one can view the set of points lying over* $\alpha^2 \in \mathbb{G}_m$ *as all points lying on two copies of* X *that are identified by the equivalence relation* $\sim$. *In particular if the functor* $\mathcal{M}_g$ *were represented by a fine moduli space* $\mathrm{M}_g$ *with a universal family* $u : \mathrm{U}_g \to \mathrm{M}_g$, *then there would be a constant map*

$$\mu_\pi : \mathbb{G}_m \to \mathrm{M}_g, \quad \alpha \mapsto [\mathrm{X}],$$

*and so* $\mathcal{F}$ *would be equal to the constant family, giving a commutative diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{F} & \mathbb{G}_m \times \mathrm{X} \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle p_1} \\ & & \mathbb{G}_m. \end{array}$$

*But the map* $F : \mathcal{F} \to \mathbb{G}_m \times \mathrm{X}$ *could simply not be well defined, for all points* $[(\alpha, p)] \in \mathcal{F}$, *and so this is impossible.*

There is a scheme $\mathrm{M}_g$ with the following properties:

1. for an algebraically closed field k, the k-points of $\mathrm{M}_g$ are in one to one correspondence with the set of isomorphism classes of smooth curves of genus g defined over k;
2. if $\pi : \mathcal{F} \to \mathrm{T}$ is a flat family of curves of genus $g$, then there is a map $\mu_\pi : \mathrm{T} \to \mathrm{M}_g$ such that if $t \in \mathrm{T}$ is a geometric point, then $\mu_\pi(t)$ is the point $[\mathcal{F}_t]$ in $\mathrm{M}_g$ corresponding to the isomorphism class of the fiber $\mathcal{F}_t = \pi^{-1}(t)$.

### 1.2.4 Coarse moduli spaces and $\overline{\mathrm{M}}_g$

**Definition 1.2.8.** *We say that a scheme* $\mathrm{Mod}_\mathcal{S}$ *is a* ***coarse moduli space*** *for the functor* $\mathcal{F}_\mathcal{S}$ *(from Definition 1.2.1), if*

1. *there is a natural transformation of functors* $\mathcal{F}_\mathcal{S} \to \mathcal{M}or_{Sch}(\quad, \mathrm{Mod}_\mathcal{S})$;
2. *the scheme* $\mathrm{Mod}_\mathcal{S}$ *is universal for* (1);
3. *for any algebraically closed field extension* $\mathrm{k} \hookrightarrow \mathrm{K}$,

   $$\mathcal{F}_\mathcal{S}(\mathrm{K}) \cong \mathcal{M}or_{Sch}(\mathrm{Spec}(\mathrm{K}), \mathrm{Mod}_\mathcal{S}) = \mathrm{Mod}_\mathcal{S}(\mathrm{K}),$$

   *is an isomorphism of sets.*

**Definition 1.2.9.** *A **stable curve** is a complete connected curve with only nodes as singularities and only finitely many automorphisms.*

**Remark 1.2.10.** *In order for a curve to have a finite number of automorphisms, any rational component must meet any other component of the curve in at least three points.*

**Definition 1.2.11.** *For $g = \dim \mathrm{H}^1(C, \mathcal{O}_C) \geqslant 2$, consider the contravariant functor:*

$$\overline{\mathcal{M}}_g : (\mathcal{S}ch_{\mathrm{k}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \overline{\mathcal{M}}_g(\mathrm{T}),$$

*where $\overline{\mathcal{M}}_g(\mathrm{T})$ is the set of flat proper morphisms $\pi : \mathcal{F} \to \mathrm{T}$ such that every fiber $\mathcal{F}_t$ is a stable curve of genus $g$ modulo isomorphism over $\mathrm{T}$.*

**Theorem 1.2.12.** *[DM69]There exists a coarse moduli space $\overline{\mathrm{M}}_g$ for the moduli functor $\overline{\mathcal{M}}_g$; Moreover, $\overline{\mathrm{M}}_g$ is a projective variety that contains $\mathrm{M}_g$ as a dense open subset.*

**Remark 1.2.13.** *Let $\mathrm{T}$ be any smooth curve and $p \in \mathrm{T}$ a (geometric) point on $\mathrm{T}$. Suppose there is a regular map*

$$\mu^* : \mathrm{T}^* = \mathrm{T} \backslash \{p\} \to \overline{\mathrm{M}}_g.$$

*By definition of coarse moduli space, this map corresponds to a family $\pi : \mathrm{X} \to \mathrm{T}^*$ of stable curves of genus $g$, parametrized by $\mathrm{T}^*$. Now by Theorem 1.2.12, the moduli space $\overline{\mathrm{M}}_g$ is proper, and so by the valuative criterion for properness, there is an extension of $\mu^*$ giving a morphism $\mu : \mathrm{T} \to \overline{\mathrm{M}}_g$. But by Theorem 1.2.12, $\overline{\mathrm{M}}_g$ is also separated, and one can use this to show this extension $\mu$ is unique. So this says that there is a unique extension to a family $\pi : \mathrm{X} \to \mathrm{T}$ parametrized by $\mathrm{T}$. This is the content of **the stable reduction theorem**.*

### 1.2.5 The boundary of $\overline{\mathrm{M}}_g$

The boundary of $\overline{\mathrm{M}}_g$ is a union of components:

$$\overline{\mathrm{M}}_g \backslash \mathrm{M}_g = \cup_{i=0}^{\lfloor \frac{g}{2} \rfloor} \Delta_i,$$

- $\Delta_0$ is the closure of the locus of curves with a single non-separating node, and
- for $i > 0$, $\Delta_i$ is the closure of the locus of curves with a single separating node whose normalization consists of a curve of genus $i$ and a curve of genus $g - i$.


Figure 1.1: Components of the boundary of $\overline{\mathrm{M}}_g$

## 1.3 $\overline{\mathrm{M}}_{g,n}$

As one can see in the images pictured in Figure 1.1, moduli of pointed curves come up naturally even if one is only interested in studying $\overline{\mathrm{M}}_g$: Each component of the boundary is the image of a morphism from a variety (or product of varieties) that (coarsely) represent a more general moduli functor

$$\overline{\mathrm{M}}_{g-1,2} \twoheadrightarrow \Delta_0, \quad \text{and for } 1 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor, \quad \overline{\mathrm{M}}_{i,1} \times \overline{\mathrm{M}}_{g-i,1} \twoheadrightarrow \Delta_i.$$

**Definition 1.3.1.** *A **stable n-pointed curve** is a complete connected curve C that has only nodes as singularities, together with an ordered collection $p_1, p_2, \ldots, p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \ldots, p_n)$ has only a finite number of automorphisms.*

**Definition 1.3.2.** *For $g = 0$, let $n \geqslant 3$, and for $g = 1$, let $n \geqslant 1$:*

$$\overline{\mathcal{M}}_{g,n} : (\mathcal{S}ch_{\mathrm{k}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \overline{\mathcal{M}}_{g,n}(\mathrm{T}),$$

*where $\overline{\mathcal{M}}_{g,n}(\mathrm{T})$ is the set of proper families $(\pi : \mathrm{X} \to \mathrm{T}; \{\sigma_i : T \to \mathrm{X}\}_{i=1}^n)$ such that the fiber $(\mathrm{X}_t, \{\sigma_i(t)\}_{i=1}^n)$, at every geometric point $t \in \mathrm{T}$ is a stable n-pointed curve of genus g modulo isomorphism over $\mathrm{T}$.*

**Theorem 1.3.3.** *[KM76, Knu83a, Knu83b] There exists a coarse moduli space $\overline{\mathrm{M}}_{g,n}$ for the moduli functor $\overline{\mathcal{M}}_{g,n}$; it is a projective variety that contains $\mathrm{M}_{g,n}$ as a dense open subset. Moreover, $\overline{\mathrm{M}}_{0,n}$ is a smooth projective variety that is a fine moduli space for $\overline{\mathcal{M}}_{0,n}$.*

**Remark 1.3.4.** *When $g = 0$, the moduli spaces $\overline{\mathrm{M}}_{0,n}$ represent the functors $\overline{\mathcal{M}}_{0,n}$, and moreover they are smooth projective rational varieties. Kapranov showed how to construct $\overline{\mathrm{M}}_{0,n}$ as both a Chow and Hilbert quotient using Veronese curves in [Kap93b], and alternatively as a Chow quotient using the Grassmannian $\mathrm{G}(2, n)$ in [Kap93a].*

### 1.3.1 Tautological maps

**Theorem 1.3.5.** *(Nodal Reduction) Let* $\mathrm{T}$ *be a smooth curve,* $p$ *a point of* $\mathrm{T}$ *and* $\mathrm{T}^* = \mathrm{T}\setminus\{p\}$*. Let* $\mathrm{X} \to \mathrm{T}^*$ *be a flat family of nodal curves of genus* $g$*,* $\psi : \mathrm{X} \to \mathrm{Z}$ *any morphism to a projective scheme* $\mathrm{Z}$*, and* $D \subset \mathrm{X}$ *any divisor finite over* $\mathrm{T}^*$*. Then there exists a branched cover* $\mathrm{T}' \to \mathrm{T}$ *and a family* $\mathrm{X}' \to \mathrm{T}'$ *of nodal curves, extending the fiber product* $\mathrm{X} \times_{\mathrm{T}^*} \mathrm{T}'$ *with the following properties:*

1. *The total space* $\mathrm{X}'$ *is smooth;*
2. *The morphism* $\pi_{\mathrm{X}} \circ \psi : \mathrm{X} \times_{\mathrm{T}^*} \mathrm{T}' \to \mathrm{Z}$ *extends to a regular morphism on all of* $\mathrm{X}'$*;*
3. *The closure of* $\pi_{\mathrm{X}}^{-1}(D)$ *in* $\mathrm{X}'$ *is a disjoint union of sections of* $\mathrm{X}' \to \mathrm{T}'$*.*

*Any two such extensions are dominated by a third and so have special fibers whose stable models are isomorphic.*

The moduli spaces $\overline{\mathrm{M}}_{g,n}$ are related to each other through tautological clutching and attaching morphisms. For example, there are

1. projection maps:
$$\pi_i : \overline{\mathrm{M}}_{g,n} \longrightarrow \overline{\mathrm{M}}_{g,n-1},$$
given by dropping the $i$-th marked point (and stabilizing, if necessary).
2. attaching maps:
$$\overline{\mathrm{M}}_{g_1,n_1+1} \times \overline{\mathrm{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathrm{M}}_{g_1+g_2,n_1+n_2},$$
given by glueing pointed curves together;
3. clutching maps:
$$c : \overline{\mathrm{M}}_{g-k,n+2k} \longrightarrow \overline{\mathrm{M}}_{g,n},$$
given by attaching marked points in pairs.

and combinations of these. It can be beneficial to think of the moduli spaces as a unified system, and ultimately many questions even about $\overline{\mathrm{M}}_g$ or as we'll see, about bundles of conformal blocks on $\overline{\mathrm{M}}_g$, can be reduced to analogous questions on $\overline{\mathrm{M}}_{0,n}$, for suitable $n$.

### 1.3.2 F-Curves on $\overline{\mathrm{M}}_{g,n}$

The moduli space $\overline{\mathrm{M}}_{g,n}$, has dimension $3g - 3 + n$ and the set of curves with at least $k$-nodes has codimension $k$. So for example,

1. $\Delta^1(\overline{\mathcal{M}}_{g,n})$ = the set of curves having at least one node. $\Delta^1$ has codimension one. This is the boundary $\overline{\mathcal{M}}_{g,n} \backslash \mathrm{M}_{g,n}$, and it is a union of irreducible components $\Delta_{g,I}$.
2. $\Delta_1(\overline{\mathcal{M}}_{g,n})$ = the set of curves having $3g-4+n$ nodes. $\Delta_1$ is 1-dimensional, composed of a union of curves whose numerical equivalence classes we call $F$-curves.

One can represent the F-curves as images of maps from $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{1,1}$. Notice that on

$\overline{\mathrm{M}}_{0,n}$ the only F-curves one has are images of maps from $\overline{\mathcal{M}}_{0,4}$. We can "keep track" of them by the partition $N = N_1 \cup N_2 \cup N_3 \cup N_4$ that defines them.

## 1.4 Cones of cycles on a projective variety

Let $X$ be a projective, not necessarily smooth variety defined over an algebraically closed field. Good references for the concepts below are Laz1,Laz2.

**Definition 1.4.1.** *A variety $X$ is called $\mathbb{Q}$-factorial if every Weil divisor on $X$ is $\mathbb{Q}$-Cartier. We assume today that $X$ is a $\mathbb{Q}$-factorial normal, projective variety over the complex numbers. The moduli spaces $\overline{\mathrm{M}}_{g,n}$ have these properties.*

**Definition 1.4.2.** *Two divisors $D_1$ and $D_2$ are numerically equivalent, written $D_1 \equiv D_2$, if they intersect all irreducible curves in the same degree. We say two curves $C_1$ and $C_2$ are numerically equivalent, written $C_1 \equiv C_2$ if $C_1 \cdot D = C_2 \cdot D$ for every irreducible subvariety $D$ of codimension one in $X$.*

**Definition 1.4.3.** *We set $N_1(X)_{\mathbb{Z}}$ equal to the vector space of curves up to numerical equivalence, and $N^1(X)_{\mathbb{Z}}$ equal to the vector space of divisors up to numerical equivalence, and set*

$$\mathrm{N}^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \;\; \mathrm{N}^1(X) = \mathrm{N}^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R},$$

*and*

$$\mathrm{N}_1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \;\; \mathrm{N}_1(X) = \mathrm{N}_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The nef and pseudo-effective cones on $X$ are subcones of vector spaces $\mathrm{N}^k(X)$, and $\mathrm{N}_k(X)$, which can be define analogously, and which can be defined for arbitrary proper varieties. This perspective involves thinking about cycles as being naturally dual to Chern classes of vector bundles.

**Definition 1.4.4.** *The pseudo effective cone* $\overline{\mathrm{Eff}}_k(X) \subset \mathrm{N}_k(\overline{\mathrm{M}}_{g,n})$ *is defined to be the closure of the cone generated by k-cycles with nonnegative coefficients. Similarly* $\overline{\mathrm{Eff}}^k(X) \subset \mathrm{N}^k(X)$ *is defined to be the closure of the cone generated by cycles of codimension k with nonnegative coefficients.*

The cones $\overline{\mathrm{Eff}}_k(X)$, and $\overline{\mathrm{Eff}}^k(X)$ are full dimensional, spanning the vector spaces $\mathrm{N}_k(X)$, and $\mathrm{N}^k(X)$. They are pointed (containing no lines), closed, and convex.

**Definition 1.4.5.** *The Nef Cone* $\mathrm{Nef}^k(X) \subset \mathrm{N}^k(X)$ *is the cone dual to* $\overline{\mathrm{Eff}}_k(X)$.

As the dual of $\overline{\mathrm{Eff}}_k(X)$, the nef cone has all of the nice properties that $\overline{\mathrm{Eff}}_k(X)$ does.

The nef cone can also be defined as the closure of the cone generated by semi-ample divisors – divisors that correspond to morphisms, and

$$f : X \to Y \text{ is a regular map, then } f^*(\mathrm{Nef}(Y)) \subset \mathrm{Nef}^1(X).$$

Given a projective variety $Y$, and a morphism $f : X \longrightarrow Y \hookrightarrow \mathbb{P}^N$, then for any ample divisor $A = \mathcal{O}(1)|_Y$ on $Y$, one has the pullback divisor $D = f^*A$ on $X$ is base point free. In fact, this divisor $D$ is not only base point free, it has the much weaker property that it is nef. For if $C$ is a curve on our projective variety $X$, then by the projection formula

$$D \cdot C = f_*(D \cdot C) = A \cdot f_*C,$$

which is zero if the map $f$ *contracts* $C$, and otherwise, as $A$ is ample, it is positive.

It is not true that every nef divisor on an arbitrary proper variety $X$ has an associated morphism; To have such a property would be very special (a dream situation). But as we saw above, the divisors that give rise to maps do live in the nef cone, and for that reason the nef cone can be used a tool to understand the birational geometry of the space.

**Definition 1.4.6.** *For a* $\mathbb{Q}$*-Cartier divisor D on a proper variety X, we define the stable base locus of D to be the union (with reduced structure) of all points in X which are in the base locus of the linear series* $|nmD|$*, for all n, where m is the smallest integer* $\geqslant 1$ *such that mD is Cartier.*

Sufficiently high and divisible multiples of any effective divisor $D$ on $X$ will define a rational map (although not necessarily a morphism) from $X$ to a projective variety $Y$. The stable base locus of $D$ is the locus where the associated rational map will not be defined. The pseudo-effective cone may be divided into chambers having to do with the stable base loci ELM1, ELM2.

A simple example illustrates how even very crude information about the location of the cone of nef divisors with respect to the effective cone tells us valuable information about the geometry of the variety $X$, as we see for $\overline{\mathrm{M}}_g$.

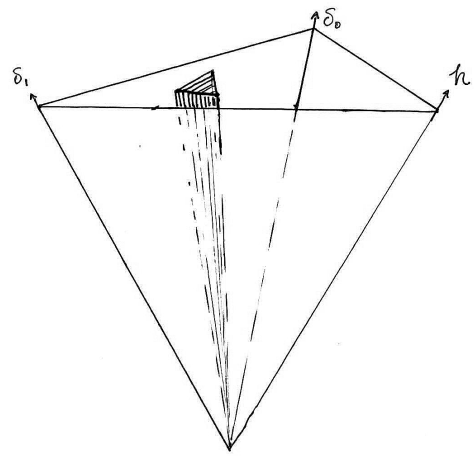



Figure 1.2: $\mathrm{Nef}^1(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$ with generators $\lambda$, $12\lambda - \delta_0$, and $10\lambda - \delta_0 - 2\delta_1$.

### 1.4.1 An example result about nef cones for $\overline{\mathrm{M}}_g$

**Theorem 1.4.7.** *Every nef divisor on $\overline{\mathrm{M}}_g$ is big. In particular, there are no morphisms, with connected fibers from $\overline{\mathrm{M}}_g$ to any lower dimensional projective varieties other than a point.*

Theorem 1.4.7 says that the nef cone of $\overline{\mathrm{M}}_g$ sits properly inside of the cone of effective divisors– and their extremal faces only touch at the origin of the Nerón Severi space.

The statement for pointed curves is a little bit more complicated, but still very simple in the grand scheme of things:

**Theorem 1.4.8.** *For $g \geqslant 2$, any nef divisor is either big or is numerically equivalent to the pullback of a big divisor by composition of projection morphisms. In particular, for $g \geqslant 2$, the only morphisms with connected fibers from $\overline{\mathrm{M}}_{g,n}$ to lower dimensional projective varieties are compositions of projections given by dropping points, followed by birational maps.*

In all the examples we know, like in the picture, the nef cones are polyhedral, and the extremal rays are generated by semi-ample divisors. One can therefore ask whether these two features hold in general:

**Question 1.4.9.**

1. *Is $\mathrm{Nef}(\overline{\mathrm{M}}_{g,n})$ polyhedral?*
2. *Is every nef divisor on $\overline{\mathrm{M}}_{g,n}$ semi-ample?*

## 1.5 Reduction of a problem for $g > 0$ to $g = 0$

In $\overline{M}_{g,n}$, the locus

$$\delta^k(\overline{M}_{g,n}) = \{(C, \vec{p}) \in \overline{M}_{g,n} \ : \ C \text{ has at least k nodes } \}$$

has codimension $k$. For each $k$, the set $\delta^k(\overline{M}_{g,n})$ decomposes into irreducible component indexed by dual graphs $\Gamma$ with $k$ edges. Moreover, the closure of the component corresponding to $\Gamma$ contains components consisting of curves whose corresponding dual graph $\Gamma'$ contracts to $\Gamma$. This gives rise to a stratification of the space which is both reminiscent and analogous to the combinatorial structure determined by the torus invariant loci of a toric variety.

On a complete toric variety, every effective cycle of dimension $k$ can be expressed as a linear combination of torus invariant cycles of dimension $k$. Fulton compared the action of the symmetric group $S_n$ on $\overline{M}_{0,n}$ with the action of an algebraic torus a toric variety. Following this analogy, he asked whether a variety of dimension $k$ could be expressed as an effective combination of boundary cycles of that dimension. As $\overline{M}_{0,n}$ is rational, of dimension $n-3$, this is true for points and cycles of codimension $n-3$. For the statement to be true for divisors, it would say that every effective divisor would be in the cone spanned by the boundary divisors. This was proved false by Keel [GKM02, page 4] and Vermeire, who found effective divisors not in the convex hull of the boundary divisors. For the statement to be true for curves, it would say that the Mori cone of curves is spanned by irreducible components of $\delta^{n-4}(\overline{M}_{0,n})$: whose dual graph is distinctive: the only vertex that isn't trivalent has valency four. In particular this says a divisor is nef if and only if it nonnegatively intersects those curves that can be described as images of attaching or clutching maps from $\overline{M}_{0,4}$.

This question could be asked for higher genus, and Faber did this independently (as an intermezzo in his thesis), proving the statement for $\overline{M}_3$ and $\overline{M}_4$.

In honor of Faber and Fulton, the numerical equivalence classes of the irreducible components of $\delta^{3g-4+n}(\overline{M}_{g,n})$ are called F-Curves. One can ask the following question:

**Question 1.5.1.** *(The* F*-Conjecture [GKM02]) Is every effective curve numerically equivalent to an effective combination of* F*-Curves? Otherwise said, is a divisor is nef, if and only if it nonnegatively intersects all the* F*-Curves?*

In [GKM02], using the flag map (see Definition 1.5.1) we showed that in fact a positive solution to this question for $S_g$-invariant nef divisors on $\overline{M}_{0,g+n}$ would give a positive answer for divisors on $\overline{M}_{g,n}$. In particular, there is the potential that the cone of nef divisors on $\overline{M}_{0,g+n}$ can tell us about the cone of nef divisors on $\overline{M}_{g,n}$. We know now that the answer to this question is true on $\overline{M}_{0,n}$ for $n \leqslant 7$ KeelMcKernan, and on $\overline{M}_g$ for $g \leqslant 24$ [Gib09].

### 1.5.1 The flag map

The flag map is defined as follows. Fix a *point* $(E, q) \in \mathrm{M}_{1,1}$ and define the morphism $f : \overline{\mathrm{M}}_{0,g+n} \longrightarrow \overline{\mathrm{M}}_{g,n}$, which takes a stable $g+n$-pointed rational curve $(C; \{q_1, \ldots, q_g\} \cup \{p_1, \ldots, p_n\})$ to a stable $n$-pointed curve of genus $g$ by attaching $g$ copies of $(E, p)$ to $C$ by gluing $C$ and $E$ by identifying $q$ and $q_i$ for $1 \leqslant i \leqslant g$. In [GKM02], we showed that an F-divisor $D$ on $\overline{\mathrm{M}}_{g,n}$ is nef if and only if $f^*D$ is nef. An F-divisor is, by definition, any divisor that nonnegatively intersects all the F-curves. Moreover, by [GKM02], every $\mathrm{S}_g$-symmetric nef divisors on $\overline{\mathrm{M}}_{0,g+n}$ is equal to the pullback of a nef divisor on $\overline{\mathrm{M}}_{g,n}$.

## 1.6 Why vector bundles of conformal blocks?

Vector bundles of covacua for affine Lie algebras give rise to elements of the cone of nef divisors: each bundle on $\overline{\mathrm{M}}_{0,n}$ is globally generated, and so has base point free first Chern class (ie. is of the form $f^*A$ for some morphism $f : \overline{\mathrm{M}}_{0,n} \to Y$ where $Y$ is a projective variety, and A is an ample line bundle on it). There are a lot of these bundles: They generate a full dimensional sub-cone of the nef cone.

The F-Conjecture, if true, would give a positive answer to Question 1.4.9 Part (1). Therefore, Question 1.4.9 and the F-Conjecture motivates our interest in vector bundles of conformal blocks. If every nef divisor on $\overline{\mathrm{M}}_{0,n}$ is a conformal blocks divisor, then the answer to Question 1.4.9 Part (2) will hold for $g = 0$. If this is true and the cone generated by conformal blocks is polyhedral, then the answer to Question 1.4.9 Part (2) is true and we have more evidence for the F-Conjecture. If If this is true and the cone generated by conformal blocks is not polyhedral, then both the answer posed by Question 1.4.9 Part (2) and the F-Conjecture are false.

Of course it may be that the nef cone is not generated by these bundles, and there is something more to the story.

There are a lot of questions, and in trying to answer just a few, we've learned new things about vector spaces of conformal blocks and the moduli space of curves, some of which I hope to share this week.

# Lecture 2

# VBs of CBs on $\overline{\mathrm{M}}_{g,n}$

My goals in this lecture are to:

1. Define vector bundles of covacua and conformal blocks on $\overline{\mathrm{M}}_{g,n}$; and
2. Discuss two important theorems
   - Factorization; and
   - Propagation of Vacua

   used in the definition, and important in almost every result obtained about the bundles, as I'll illustrate throughout the remaining lectures.

If there is time, I'd like to say something about computing the ranks of these bundles.

Vector bundles of conformal blocks $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ on $\overline{\mathcal{M}}_{g,n}$ are determined by collections of data including:

- a simple Lie algebra $\mathfrak{g}$;
- a positive integer $\ell$; and
- an $n$-tuple $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of dominant integral weights for $\mathfrak{g}$ at level $\ell$;

For the bundles to have nontrivial rank, the triples should satisfy a compatibility criterion, which will be described.

In my lecture I assumed that members of the audience would look up the meaning of these terms from representation theory if they were unfamiliar. There are many other better references (for instance [FH91]), but for convenience, I have included basic definitions here in these notes. After defining these terms in Section 2.1, I will describe these bundles in Section 2.2.

## 2.1 Just enough representation theory

We begin by describing the ingredients that go into the definitions of the bundles.

### 2.1.1 Simple Lie algebras

Throughout, we fix a field $k$, which will be useful to assume later is algebraically closed, and of characteristic 0.

**Definition 2.1.1.** *A **Lie algebra** is a k-vector space $\mathfrak{g}$ together with an binary operation called **the Lie bracket***

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (A, B) \to [A, B]$$

*which satisfies the following three conditions*

1. *bilinearity:* $[A + B, C] = [A, C] + [B, C]$ *and* $[A, B + C] = [A, B] + [A, C]$;
2. *anti-symmetry:* $[A, A] = 0$; *or equivalently if* $\mathrm{char}(k) \neq 2$, $[A, B] = -[B, A]$; *and*
3. *the Jacobi identity:* $[A, [B, C]] - [[A, B], C] = [B, [A, C]]$.

**Example 2.1.2.** *Let $V$ be a k-vector space of dimension $n$. We let $gl(V)$ be the **general linear Lie algebra**, consisting of the set of linear transformations $V \to V$, and Lie bracket given by the commutator $[\phi, \theta] = \phi \circ \theta - \theta \circ \phi$.*

*In particular, as is conventional, we denote $gl(k^n)$ by $gl_n$, taking elements to be $n \times n$ matrices over $k$, and the Lie bracket to be the commutator:*

$$[A, B] = AB - BA.$$

*Clearly this is bilinear and anti-symmetric. One may also verify that the Jacobi identity:*

$$\begin{aligned}(2.1)\quad [A, [B, C]] - [[A, B], C] &= \big(A(BC - CB) - (BC - CB)A\big) - \big((AB - BA)C - C(AB - BA)\big)\\ &= ABC - ACB - BCA + CBA + ABC + BAC + CAB - CBA\\ &= BAC + CAB - ACB - BCA = [B, [A, C]].\end{aligned}$$

**Definition 2.1.3.** *A Lie algebra $\mathfrak{g}$ is **Abelian** if $[A, B] = 0$ for every $A, B \in \mathfrak{g}$.*

**Definition 2.1.4.** *A Lie algebra is* **simple** *if it is not Abelian, and has no nonzero proper ideals.*

### 2.1.2 Dominant integral weights for $\mathfrak{g}$

To define dominant integral weights for $\mathfrak{g}$ we start with representations of $\mathfrak{g}$.

**Definition 2.1.5.** *A* **homomorphism of Lie algebras** *is a linear map of vector spaces $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ preserving the bracket:*

$$f(\, [A, B]_{\mathfrak{g}_1}) = [f(A), f(B)]_{\mathfrak{g}_2},\ \forall\, A, B \in \mathfrak{g}_1.$$

**Definition 2.1.6.** *Let $V$ be a vector space, and $\mathfrak{g}$ a Lie algebra. A* ***representation*** *of $\mathfrak{g}$ on $V$ is a Lie algebra homomorphism $\mathfrak{g} \to gl(V)$. Equivalently, a representation of $\mathfrak{g}$ on $V$ is a rule $\mathfrak{g} \times V \to V$, say $(A, v) \mapsto A \cdot v$ such that*

$$[A, B] \cdot v = A \cdot (B \cdot v) - B \cdot (A \cdot v).$$

**Remark 2.1.7.** *If $\mathfrak{g} \to gl(V)$ is a representation of $\mathfrak{g}$ on $V$, we often abuse language and simply refer to $V$ itself as a representation (omitting the homomorphism from the notation).*

**Definition 2.1.8.** *If $\mathfrak{g}$ is a Lie algebra, then it acts on itself via*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (A, B) \mapsto A \cdot B = [A, B].$$

*This gives the homomorphism of Lie algebras*

$$ad_{\mathfrak{g}} : \mathfrak{g} \to gl(\mathfrak{g}), \quad A \to ad_{\mathfrak{g}}(A),$$

*where $ad_{\mathfrak{g}}(A)$ is the linear transformation on defined by*

$$ad_{\mathfrak{g}}(A)(B) = [A, B].$$

*This very important representation is referred to as* ***the adjoint representation****.*

**Definition 2.1.9.** *We say that a representation of $\mathfrak{g}$ on $V$ is* ***irreducible*** *if it has no nontrivial proper sub-representations. That is, if there is no non-trivial and proper vector subspace $W \subset V$ and representation $\mathfrak{g} \to gl(W)$, making the natural induced diagram:*

$$\mathfrak{g} \to gl(W) \subset gl(V),$$

*commute.*

**Definition 2.1.10.** *A linear subspace $\mathfrak{g}_1 \subset \mathfrak{g}_2$ is a* **Lie subalgebra** *if $\mathfrak{g}_1$ is closed under the Lie bracket of $\mathfrak{g}_2$:*

$$[A, B]_{\mathfrak{g}_2} \in \mathfrak{g}_1,\ \forall\, A, B \in \mathfrak{g}_1.$$

If $\mathfrak{g} \to gl(W)$ is a sub-representation of $V$, then $gl(W) \subset gl(V)$ is a Lie subalgebra.

**Example 2.1.11.** *Let $sl(V)$ (resp. $sl_n$) denote the Lie subalgebra of $gl(V)$ (resp. $gl_n$) called the* ***special linear Lie algebra*** *consisting of those operators on $V$ of trace $0$ (ie. those matrices whose trace is $0$).*

**Definition 2.1.12.** *A* ***Cartan subalgebra*** *of a Lie algebra $\mathfrak{g}$ is an Abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is maximal with respect to the property of being Abelian.*

**Exercise 2.1.13.** *Let $\mathfrak{h} \subset sl_n$ be the diagonal matricies. Show $\mathfrak{h}$ is a Cartan subalgebra.*

**Definition 2.1.14.** *Let $\mathfrak{g}$ be a Lie algebra and $V$ be a representation for $\mathfrak{g}$. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. We describe the* **weights and roots for** $\mathfrak{g} \to gl(V)$ *as follows:*

1. *By an* ***eigenvalue*** *for the action of $\mathfrak{h}$, we will mean an element $\alpha \in \mathfrak{h}^*$ such that $H(v) = \alpha(H) \cdot v$, for some nonzero $v \in V$, and all $H \in \mathfrak{h}$. An eigenvalue $\alpha \in \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on the representation $V$ of $\mathfrak{g}$ is called a* ***weight of the representation****. The weights $\alpha \in \mathfrak{h}^*$ that occur in the adjoint representation are called* ***roots****. The convention is that $0 \in \mathfrak{h}^*$ is not considered a root.*
2. *By the* ***eigenspace*** *$V_\alpha$ associated to the eigenvalue $\alpha$ we mean the subspace of all vectors $v \in V$ such that $H(v) = \alpha(H) \cdot v$. The corresponding eigenvectors in $V_\alpha$ are called* ***weight vectors*** *and $V_\alpha$ is called the* ***weight space****. The eigenspaces $\mathfrak{g}_\alpha$ corresponding to the roots are called* ***root spaces****.*

**Definition 2.1.15.** *We define the* **weights and roots for** $\mathfrak{g}$ *as follows.*

1. *The weights for $\mathfrak{g}$ are the weights for all representations $\mathfrak{g} \to gl(V)$.*
2. *We denote the set of all roots by $R \subset \mathfrak{h}^*$.*

**Definition 2.1.16.** *One can define a* **highest weight** *as follows:*

- *We choose a direction in $\mathfrak{h}^*$ which means defining a linear functional $f : \mathfrak{h}^* \to \mathbb{C}$. This gives a decomposition of the set*

  $$R = R^+ \cup R^-, \text{ where}$$

  *$R^+ = \{\alpha \in R : f(\alpha) > 0\}$, called the* ***positive roots****, and*
  *$R^- = \{\alpha \in R : f(\alpha) < 0\}$, called the* ***negative roots****.*
- *We say that a positive (resp., negative) root $\alpha \in R$ is primitive or* ***simple*** *if it cannot be expressed as a sum of two positive (resp. negative) roots.*

– *A nonzero vector $v \in V$ which is both an eigenvector for the action of $\mathfrak{h}$ and in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$ is called a* ***highest weight vector****.*

**Remark 2.1.17.** *In Definition 2.1.24 we will describe the Killing form. After that we will be able to define a semisimple Lie algebra over a field of characteristic zero as one whose Killing form is nondegenerate. The following can be shown to be equivalent for a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0:*

1. *$\mathfrak{g}$ is semisimple;*
2. *$\mathfrak{g}$ is a finite direct product of simple Lie algebras.*

*In particular, if $\mathfrak{g}$ is a finite dimensional simple Lie algebra defined over a field of characteristic 0, then $\mathfrak{g}$ is semisimple. While not necessary for our application, the next statement holds for the broader context of semisimple Lie algebras.*

**Proposition 2.1.18.** *[FH91, 14.13] For any semisimple complex Lie algebra $\mathfrak{g}$,*

1. *every finite dimensional representation $V$ of $\mathfrak{g}$ has a highest weight vector;*
2. *an irreducible representation has a unique highest weight vector up to scalars.*

**Definition 2.1.19.** *A* ***dominant integral weight*** *is an element $\alpha \in \mathfrak{h}^*$ such that $H(v) = \alpha(H) \cdot v$, for all $H \in \mathfrak{h}$, where $v \in V$ is the highest weight vector of an irreducible representation $V$ of $\mathfrak{h}$.*

**Definition 2.1.20.** *[FH91, Section 14.2] $R$ generates a lattice $\Lambda_{\mathbf{R}} \subset \mathfrak{h}^*$, called the* ***root lattice****, of rank equal to $\dim(\mathfrak{h})$. The free generators for the lattice are called* ***fundamental dominant weights****.*

**Remark 2.1.21.** *Depending on the author, weights are sometimes called integral weights; dominant integral weights are sometimes referred to as dominant weights.*

**Definition 2.1.22.** *A character of a Lie algebra $\mathfrak{g}$ is a linear map $\mathfrak{g} \to k$. That is, since $k = gl_1$, a character of a Lie algebra $\mathfrak{g}$ is a 1-dimensional representation of $\mathfrak{g}$.*

**Example 2.1.23.** *Let $\mathfrak{g} = sl_2$. We first set*

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}.$$

*Then*

$$ad_{\mathfrak{g}}(A) : sl_2 \longrightarrow sl_2, \;\; B \mapsto AB - BA,$$

*so that in particular*

$$ad_{\mathfrak{g}}(A)\begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} bz - yc & 2(ay - bx) \\ 2(cx - az) & -(bz - yc) \end{pmatrix}.$$

*The Cartan subalgebra* $\mathfrak{h}$ *is the set of diagonal matrices in* $\mathfrak{g} = sl_2$*. Consider*

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{h},$$

*so that*

$$ad_{\mathfrak{g}}(A) : sl_2 \to sl_2, \quad \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mapsto 2a \begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix}.$$

*We shall see that* $ad_{\mathfrak{g}}(A)$ *is a direct sum of three characters of* $\mathfrak{h}^*$*. Namely, one can decompose* $sl_2$ *as a direct sum of three one-dimensional vector spaces* $sl_2 \cong V_1 \oplus V_2 \oplus V_3$*, where*

$$V_1 = \{\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x \in k\}; \quad V_2 = \{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in k\};$$

*and*

$$V_3 = \{\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} : z \in k\}.$$

*The sub-vector spaces* $V_i \subset sl_2$ *are sub-representations of the adjoint representation of* $\mathfrak{h}$ *on* $sl_2$ *defined by*

$$\mathfrak{h} \times V_1 \to V_1, \ (\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

$$\mathfrak{h} \times V_2 \to V_2, \ (\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}) \mapsto 2a \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix};$$

$$\mathfrak{h} \times V_3 \to V_3, \ (\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}) \mapsto -2a \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}.$$

*The second and third characters* $\alpha_1 = 2a$ *and* $\alpha_2 = -2a$*, which are the nonzero representations, are the two roots on* $sl_2$*. The root* $\alpha_1$ *is a simple root. In general, one has* $r$ *simple roots of* $sl_{r+1}$*.*

### 2.1.3 Dominant integral weights for $\mathfrak{g}$ at level $\ell$

In order to define the level of a weight, we next define the Killing form $(\,|\,)$, and the normalized Killing form $(\,,\,)$, which both come from an inner product $<\ |\ >$ on $\mathfrak{g}$.

**Definition 2.1.24.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Recall that for $A \in \mathfrak{g}$, one has the adjoint representation*

$$ad_{\mathfrak{g}}(A) : \mathfrak{g} \to \mathfrak{g}, \;\; C \mapsto ad_{\mathfrak{g}}(A)(C) = [A, C].$$

*In particular, a choice of basis for $\mathfrak{g}$ gives a representation of this linear transformation* $\mathrm{ad}_{\mathfrak{g}}(A)$ *by a square matrix of* $\dim(\mathfrak{g})$. *We define an inner product $<\,|\,>$ on $\mathfrak{g}$ by setting, for $A$ and $B \in \mathfrak{g}$,*

$$< A \mid B >= \mathrm{trace}(\mathrm{ad}_{\mathfrak{g}}(A) \cdot \mathrm{ad}_{\mathfrak{g}}(B)).$$

*One can then define a natural morphism from $\mathfrak{h}$ to $\mathfrak{h}^*$ by setting*

$$\psi : \mathfrak{h} \to \mathfrak{h}^* = Hom(\mathfrak{h}, k), \;\; A \mapsto \{B \mapsto < A \mid B >\}.$$

*One can check that this is an isomorphism and that this induces an inner product on $\mathfrak{h}^*$:*

$$(f|g) :=< \psi^{-1}(f) \mid \psi^{-1}(g) >= trace(ad_{\mathfrak{g}}(\psi^{-1}(f)) \cdot ad_{\mathfrak{g}}(\psi^{-1}(g))).$$

*This natural inner product $(\,|\,)$ is referred to as the* ***Killing form****.*

**Remark 2.1.25.** *One can prove that there is a unique positive root $\theta \in R^+$ with the property that $(\theta|\theta) \geqslant (\alpha|\alpha)$ for any other root $\alpha \in R^+$. This root theta is called the* **longest root**. *It is conventional to normalize the Killing form, writing $(\,,\,)$, so that $(\theta, \theta) = 2$.*

**Definition 2.1.26.** *The* **level** *of any weight $\alpha$ is equal to the value $(\alpha, \theta)$, where $\theta$ is the longest root, and $(\,,\,)$ is the normalized Killing form.*

**Remark 2.1.27.** *The level of a dominant integral weight $\alpha$ is an integer.*

## 2.2 Definition of the fibers of the bundles

Let $\mathfrak{g}$ be a simple Lie algebra, and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be $n$ dominant integral weights for $\mathfrak{g}$ at level $\ell$. To begin, we describe a fiber of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ on $\overline{\mathrm{M}}_{g,n}$ at a point $(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n}$, such that $U = C \backslash \{p_1, \ldots, p_n\}$ is affine.

This always happens for example in case $C$ is a smooth curve of genus $g$ with at least one marked point (ie. when $n \geqslant 1$), but can also happen if there is at least one marked point on each component of a stable curve $C$ with singularities.

In case $U = C \backslash \{p_1, \ldots, p_n\}$ is not affine, we will add a marked point on every component together with a zero weight for every added point, and then use the Propagation of vacua theorem to finish the construction.

So for now we assume that $U = C \backslash \{p_1, \ldots, p_n\}$ is affine.

**We first consider a finite dimensional situation:**

As explained earlier in the lecture, to each such weight $\lambda_i$ there corresponds a unique finite dimensional $\mathfrak{g}$-module $V_{\lambda_i}$.

Set $\mathrm{V}_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, and define an action

$$\mathfrak{g} \times \mathrm{V}_{\vec{\lambda}} \to \mathrm{V}_{\vec{\lambda}}, \quad (g, v_1 \otimes \cdots v_n) \mapsto \sum_{i=1}^{n} v_1 \otimes \cdots v_{i-1} \otimes (g \cdot v_i) \otimes v_{i+1} \otimes \cdots v_n.$$

We write $[\mathrm{V}_{\vec{\lambda}}]_{\mathfrak{g}}$ for the **space of coinvariants of** $\mathrm{V}_{\vec{\lambda}}$: The largest quotient of $\mathrm{V}_{\vec{\lambda}}$ on which $\mathfrak{g}$ acts trivially. That is, the quotient of $\mathrm{V}_{\vec{\lambda}}$ by the subspace spanned by the vectors $\mathrm{X} \cdot v$ where $\mathrm{X} \in \mathfrak{g}$ and $v \in \mathrm{V}_{\vec{\lambda}}$.

The fibers $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}$ are also vector spaces of coinvariants, analogous to $[\mathrm{V}_{\vec{\lambda}}]_{\mathfrak{g}}$, only they have something to do with the point $(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n}$, as we next explain.

**Infinite dimensional analogues:**

Given a smooth n-pointed curve $(C, \vec{p})$, to construct the fiber $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}$ we will use two new Lie algebras:

**First Lie algebra:** For each $i \in \{1, \ldots, n\}$ we set $\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C} \cdot c$, where by $\mathbb{C}((\xi_i))$, we mean the field of Laurant power series over $\mathbb{C}$ in the variable $\xi_i$, and $c$ is in the center of $\hat{\mathfrak{g}}_i$. To define the bracket, we note that elements in $\hat{\mathfrak{g}}_i$ are tuples $(a_i, \alpha c)$, with $a_i = \sum_j \mathrm{X}_{ij} \otimes f_{ij}$, with $f_{ij} \in \mathbb{C}((\xi_i))$. We define the bracket on simple tensors:

$$[(\mathrm{X} \otimes f, \alpha c), (\mathrm{Y} \otimes g, \beta c)] = ([\mathrm{X}, \mathrm{Y}] \otimes fg, c(\mathrm{X}, \mathrm{Y}) \cdot \mathrm{Res}_{\xi_i = 0}(g(\xi_i) df(\xi_i))).$$

Checking $\hat{\mathfrak{g}}_i$ is a Lie algebra done in Section 2.1, where we also outline the construction of the infinite dimensional analogue $\mathrm{H}_{\lambda_i}$ of $\mathrm{V}_{\lambda_i}$: It turns out that $\mathrm{H}_{\lambda_i}$ is a unique $\hat{\mathfrak{g}}_i$-module, although infinite dimensional.

**Second Lie algebra:**

Let $U = C \setminus \{p_1, \dots, p_n\}$. Because $C$ is smooth, and has at least one marked point, $U$ is affine. By $\mathfrak{g}(U)$ we mean the Lie algebra $\mathfrak{g} \otimes \mathcal{O}_C(U)$.

Choose a local coordinate $\xi_i$ at each point $p_i$, and denote by $f_{p_i}$ the Laurant expansion of any element $f \in \mathcal{O}_C(U)$. Then for each $i$, we get a ring homomorphism

$$\mathcal{O}_C(U) \to \mathbb{C}((\xi_i)), \ \ f \mapsto f_{p_i},$$

and hence for each $i$, we obtain a map (this is not a Lie algebra embedding)

$$\mathfrak{g}(U) \to \hat{\mathfrak{g}}_i \ \ X \otimes f \mapsto (X \otimes f_{p_i}, 0).$$

Set $H_{\vec{\lambda}} = H_{\lambda_1} \otimes \cdots H_{\lambda_n}$ and define the following, which we will show is an action:
(2.2)

$$\mathfrak{g}(U) \times H_{\vec{\lambda}} \to H_{\vec{\lambda}} \ \ (X \otimes f, w_1 \otimes \cdots w_n) \mapsto \sum_{i=1}^{n} w_1 \otimes \cdots w_{i-1} \otimes ((X \otimes f_{p_i}) \cdot w_i) \otimes w_{i+1} \otimes \cdots w_n.$$

**Claim 2.2.1.** *Equation 2.2 defines an action of* $\mathfrak{g}(U)$ *on* $H_{\vec{\lambda}}$.

*Proof.* Given $X \otimes f$, and $Y \otimes g \in \mathfrak{g}(U)$, and a simple tensor $v = v_1 \otimes \cdots \otimes v_n \in H_{\vec{\lambda}}$, we want to check that

$$[X \otimes f, Y \otimes g] \cdot v = (X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big).$$

The right hand side simplifies as follows:

$$
\begin{aligned}
(2.3)\quad & (X\otimes f)\cdot\big((Y\otimes g)\cdot v\big)-(Y\otimes g)\cdot\big((X\otimes f)\cdot v\big)\\
&= (X\otimes f)\cdot\Big(\sum_{i=1}^{n} v_1\otimes\cdots\otimes v_{i-1}\otimes(Y\otimes g_{p_i})\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&\quad-(Y\otimes g)\cdot\Big(\sum_{i=1}^{n} v_1\otimes\cdots\otimes v_{i-1}\otimes(X\otimes f_{p_i})\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&=\Big(\sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} v_1\otimes\cdots v_{j-1}\otimes(X\otimes f_{p_j})\cdot v_j\otimes v_{j+1}\otimes\cdots\otimes v_{i-1}\otimes(Y\otimes g_{p_i})\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&\quad-\Big(\sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} v_1\otimes\cdots v_{j-1}\otimes(Y\otimes g_{p_j})\cdot v_j\otimes v_{j+1}\otimes\cdots\otimes v_{i-1}\otimes(X\otimes f_{p_i})\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&=\Big(\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots v_{j-1}\otimes\cdots\otimes v_{i-1}\otimes(X\otimes f_{p_i})\cdot\big((Y\otimes g_{p_i})\cdot v_i\big)\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&\quad-\Big(\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots v_{j-1}\otimes\cdots\otimes v_{i-1}\otimes(Y\otimes g_{p_i})\cdot\big((X\otimes f_{p_i})\cdot v_i\big)\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&=\Big(\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots v_{j-1}\otimes\cdots\otimes v_{i-1}\otimes\big([X,Y]+(fg)_{p_i}\big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)
\end{aligned}
$$

The left hand side simplifies as follows:

(2.4)

$$
\begin{aligned}
&\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes\Big([X,Y]\otimes f_{p_i}g_{p_i}+(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}c\Big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\\
&\qquad=\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes\Big([X,Y]\otimes f_{p_i}g_{p_i}\Big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\\
&\qquad\quad+\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes\Big((X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}c\Big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n.
\end{aligned}
$$

Now, by definition, $c\cdot v_i=\ell\cdot v_i$ for all $i$, and so we can rewrite the second summand as follows

$$
\begin{aligned}
(2.5)\quad &\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes((X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}c)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\\
&=\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}\Big(v_1\otimes\cdots\otimes v_{i-1}\otimes c\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&=\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}\Big(v_1\otimes\cdots\otimes v_{i-1}\otimes \ell\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)\\
&=\big(\ell\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}\big)\Big(v_1\otimes\cdots\otimes v_n\Big).
\end{aligned}
$$

Since $\sum_{1\leqslant i\leqslant n}(X,Y)\,\mathrm{Res}_{\xi_i=0}\, g_{p_i} df_{p_i} = 0$, this contribution is zero. Therefore the left and right hand sides of the expressions are the same, and we have checked that $\mathfrak{g}(U)$ acts on $\mathrm{H}_{\vec{\lambda}}$ as claimed. □

We now set

$$\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(C,\vec{p})} = [\mathrm{H}_{\vec{\lambda}}]_{\mathfrak{g}(U)}.$$

**To define the fiber in case the set $U$ is not affine:** We use use a result called "Propagation of Vacua", described in Section 2.4.

**If $C$ is a stable curve with singularities:** The "Factorization Theorem", described in the lecture, and in Section 2.3, gives a way to view fibers at points where curves have singularities.

Propagation of Vacua and Factorization are fundamental results.

Propagation of vacua and Beauville's quotient construction are both consequences of Theorem 2.6.1. Propagation of Vacua enables one to express a bundle with a zero weight on $\overline{\mathcal{M}}_{g,n}$ as the pullback of a bundle from $\overline{\mathcal{M}}_{g,n-1}$. Beauville's quotient construction gives an alternative expression for vector spaces of covacua in genus zero.

The Factorization Theorem, originally proved by Tsuyshiya, Ueno and Yamada [TUY89, Prop 2.2.6], explains how a vector bundle of conformal blocks at a point on the moduli space where the underlying curve has a node, factors into sums and products of bundles on the normalization of the curve where the sum is taken over all possible weights at points over which the normalization is "glued" to make the original curve. Applications of Factorization include inductive formulas for the rank and Chern classes of the bundle. In fact, Beauville, in [Bea96] gives an elementary proof of Factorization using this quotient construction.

## 2.3 Factorization

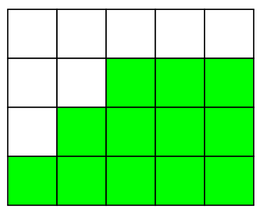
**Definition 2.3.1.** *Given a weight $\mu \in \mathcal{P}_\ell(\mathfrak{g})$, let $\mu^\star \in \mathcal{P}_\ell(\mathfrak{g})$ be the element with the property that $-\mu^\star$ is the lowest weight of the weight space $V_\mu$.*

**Example 2.3.2.** *If $\mu \in \mathcal{P}_\ell(\mathfrak{sl}_2)$, then $\mu^\star = \mu$.*

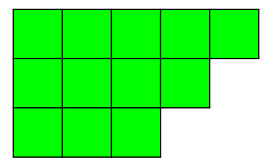
**Example 2.3.3.** *For $\mathfrak{g} = \mathfrak{sl}_{r+1}$ we express a weight $\lambda_i$ as a sum $\lambda_i = \sum_{j=1}^{r} c_j\omega_j$, and $\lambda_i$ has a corresponding Young diagram that fits into an $(r+1)\times\ell$ sized grid, where since $\lambda_i$ is* *normalized, the last row is empty. In terms of Young diagrams, the level is the number of "filled in" boxes across the top, and $|\lambda_i|$ means the total number of boxes "filled in" altogether. To find the Young diagram corresponding to $\lambda^\star$ we fill in the boxes in the diagram directly below the boxes corresponding to $\lambda$, and then rotate by 180 degrees to get the Young diagram associated to the weight $\lambda^*$. For example, if $r+1=4$, and $\ell \geqslant 5$ for the weight $\lambda$ pictured in white on the left below, then the dual weight $\lambda^\star$ is pictured in green on the right.*

Figure 2.1

$\lambda = 3\omega_1 + \omega_2 + \omega_3$
for $\mathfrak{sl}_4$,
and level $\ell(\lambda) \geqslant 5$.



$\lambda^\star = \omega_1 + \omega_2 + 3\omega_3$.



**Theorem 2.3.4** (Factorization)**.** *Let $(C_0; p_1, \ldots, p_n)$ be a stable $n$-pointed curve of genus $g$ where $C_0$ has a node $x_0$.*

1. *If $x_o$ is a non-separating node, $\nu : C \to C_0$ the normalization of $C_0$ at $x_0$, and $\nu^{-1}(x_0) = \{x_1, x_2\}$, then*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \mu \cup \mu^\star\}, \ell)_{(C;\vec{p} \cup \{x_1, x_2\})}.$$

2. *If $x_0$ is a separating node, $\nu : C_1 \cup C_2 \to C_0$ the normalization of $C_0$ at $x_0$ and $\nu^{-1}(x_0) = \{x_1, x_2\}$, with $x_i \in C_i$, then*

$$\begin{aligned} &(2.6) \quad \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \\ &\cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \lambda(C_1) \cup \{\mu\}, \ell)|_{(C_1;\{p_i \in C_1\} \cup \{x_1\})} \otimes \mathbb{V}(\mathfrak{g}, \lambda(C_2) \cup \{\mu^\star\}, \ell)|_{(C_2;\{p_i \in C_2\} \cup \{x_2\})}, \end{aligned}$$

*where $\lambda(C_i) = \{\lambda_j | p_j \in C_i\}$.*

**Definition 2.3.5.** *The weights $\mu$ and $\mu^* \in \mathcal{P}_\ell(\mathfrak{g})$ that occur in Theorem 2.3.4 are called the restriction data for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ at the point $(C_0; \vec{p})$.*

**Example 2.3.6.** *[BGM13] We will factorize the bundle $\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)$ on $\overline{\mathcal{M}}_{0,4}$ at the two types of points $(C; p_1, \ldots, p_4)$, where the curve $C$ has one node: the first* *type* $X_1 = (C_{11} \cup C_{12}; p_1, \dots, p_4)$ *where* $C_{11}$ *is labeled by* $p_1$ *and* $p_2$ *and* $C_{12}$ *by* $p_3$ *and* $p_4$*; and the second type of curve* $X_2 = (C_{21} \cup C_{22}; p_1, \dots, p_4)$ *where* $C_{21}$ *is labeled by* $p_1$ *and* $p_3$ *and* $C_{22}$ *by* $p_3$ *and* $p_4$*.*

1. *If* $r+1 = 2$ *this is* $\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_1\}, \ell)$*, and we obtain:*

$$\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_r\}, \ell)|_{X_1}$$
$$\cong \bigoplus_{\substack{m \geqslant 0 \\ even}} \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, m\omega_1\}, \ell)|_{(C_{11}, p_1, p_2, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_2, \{\ell\omega_1, \ell\omega_r, m\omega_1\}, \ell)|_{(C_{12}, p_3, p_4, x_2)}.$$

*As we'll see later, the only term in the sum above that gives bundles of nonzero rank occurs when* $m = 0$*, and that both bundles have rank one.*

$$\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_r\}, \ell)|_{X_2}$$
$$\cong \bigoplus_{\substack{m \geqslant 0 \\ m+\ell \equiv 1 (\mathrm{mod}\, 2)}} \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \ell\omega_1, m\omega_1\}, \ell)|_{(C_{21}, p_1, p_3, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \ell\omega_1, m\omega_1\}, \ell)|_{(C_{22}, p_2, p_4, x_2)}.$$

*Again, we'll see that the only term above that gives two bundles of nonzero rank occurs when* $m = (\ell - 1)$*, and has rank one in this case.*

2. *If* $r+1 = 3$ *this is* $\mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_2, \ell\omega_1\}, \ell)$*, and we obtain, for*

$$\mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_2, \ell\omega_1\}, \ell)|_{X_1}$$
$$\cong \bigoplus_{\substack{\mu = c_1\omega_1 + c_2\omega_2 \\ c_1 + 2c_2 \equiv 1 (\mathrm{mod}\, 3)}} \mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, \mu\}, \ell)|_{(C_{11}, p_1, p_2, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_3, \{\ell\omega_1, \ell\omega_r, \mu^\star\}, \ell)|_{(C_{12}, p_3, p_4, x_2)}.$$

*We'll later see that the only summand on the right hand side with nonzero rank is the one with* $\mu = \omega_1$ *(so* $c_1 = 1$*, and* $c_2 = 0$*).*

$$\mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_2, \ell\omega_1\}, \ell)|_{X_2} \tag{2.7}$$
$$\cong \bigoplus_{\substack{\mu = c_1\omega_1 + c_2\omega_2 \\ \ell + c_1 + 2c_2 \equiv 1 (\mathrm{mod}\, 3)}} \mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \ell\omega_1, \mu\}, \ell)|_{(C_{21}, p_1, p_3, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \ell\omega_2, \mu^\star\}, \ell)|_{(C_{22}, p_2, p_4, x_2)}.$$

*We'll later see that the only summand on the right hand side with nonzero rank is the one with* $\mu = (\ell-1)\omega_2$ *(so* $c_1 = 0$*, and* $c_2 = (\ell - 1)$*).*

3. *In general:*

$$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)|_{X_1}$$
$$\cong \bigoplus_{\substack{\mu = \sum_{i=1}^r c_i\omega_i \\ \sum_{i=1}^r i \cdot c_i + 2 \equiv 0 (\mathrm{mod}(r+1))}} \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, \mu\}, \ell)|_{(C_{11}, p_1, p_2, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_{r+1}, \{(\ell-1)\omega_1 + \omega_r, \ell\omega_r, \mu^\star, \ell).$$

*Moreover, one can show that the only summand on the right hand side with nonzero rank is the one with* $\mu = \omega_{r-1}$.

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)|_{X_2}\\ &\cong \bigoplus_I \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, (\ell-1)\omega_1 + \omega_r, \mu\}, \ell)|_{(C_{21}, p_1, p_3, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \ell\omega_r, \mu^\star, \ell)|_{(C_{22}, p_2, p_4, x_2)},\end{aligned}$$

*where we sum over the set*

$$I = \{\mu = \sum_{i=1}^{r} c_i\omega_i \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1}) : \sum_{i=1}^{r} i \cdot c_i + \ell + r \equiv 0(\mathrm{mod}(r+1))\}.$$

*We will eventually show that the only summand on the right hand side with nonzero rank is the one with* $\mu = (\ell-1)\omega_r$ *and* $\mu^\star = (\ell-1)\omega_1$. *We'll see that:*

$$\mathrm{rk}\, \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, (\ell-1)\omega_1 + \omega_r, (\ell-1)\omega_r\}, \ell) = \mathrm{rk}\, \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \ell\omega_r, (\ell-1)\omega_1, \ell) = 1.$$

**Remark 2.3.7.** *This example exhibits the potential for the use of factorization to compute ranks, which is the idea behind the proof of the Verlinde formula. The comments made also indicate that there is a lot of vanishing happening – which is a foreshadowing of one of the open problems in the subject: that is to determine given* $\mathfrak{g}$ *and* $\ell$ *necessary and sufficient conditions which will guarantee that the first Chern class of the bundle* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *is not zero. One indication is that it's rank is nonzero, which is actually enough for* $\mathfrak{sl}_2$, *but this is not in general. For example, while the rank of* $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3\}, 3)$ *is one, the first Chern class of this bundle is zero [BGM15a]. We'll discuss this problem.*

## 2.4 Propagation of Vacua

Propagation of Vacua is a Corollary of Theorem 2.6.1, proved in Section 2.6.

**Corollary 2.4.1.** *[Propagation of Vacua] Let* $q \in C \backslash \vec{p}$. *There is a canonical isomorphism*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C, \vec{p})} \cong \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \{0\}, \ell)|_{(C, \vec{p} \cup \{q\})}.$$

*Proof.* (of Corollary 2.4.1) Apply Theorem 2.6.1 with $\{q_1, \ldots, q_t\} = \{q\}$, and $\{\mu_1, \ldots, \mu_t\} = \{\mu = 0\}$, using that $V_0 = 0$.

□

## 2.5 Brief sketch of construction of the sheaf of conformal blocks

Suppose that $k$ is an algebraically closed field of characteristic 0. Given a triple $(\mathfrak{g}, \vec{\lambda}, \ell)$ as above, for each $i \in \{1, \ldots, n\}$, let

$$\hat{\mathfrak{g}}_i = \big(\mathfrak{g} \otimes k((\xi_i))\big) \oplus k \cdot c,$$

be the affine Lie algebra with bracket

$$[X \otimes f(\xi_i), Y \otimes g(\xi_i)] = [X, Y] \otimes f(\xi_i) g(\xi_i) + (X, Y) \cdot \operatorname{Res}(g(\xi_i) df(\xi_i)) \cdot c,$$

where $X, Y \in \mathfrak{g}$, and $c$ is in the center of $\hat{\mathfrak{g}}_i$.

As before, for each $\lambda_i$, there is a unique $\hat{\mathfrak{g}}_i$-module $H_{\lambda_i}$. Set $H_{\vec{\lambda}} = \bigotimes_{i=1}^{n} H_{\lambda_i}$, and let $T$ be a smooth variety over a field $k$, and $\pi : C \to T$ a proper flat family of curves whose fibers have at worst ordinary double point singularities. For $1 \leqslant i \leqslant n$, let $p_i : T \to C$ be sections of $\pi$ whose images are disjoint and contained in the smooth locus of $\pi$.

First, suppose that $T = \operatorname{Spec}(A)$ for A a $k$-algebra, and assume for each $i$, there are isomorphisms $\eta_i : \widehat{O}_{C,p_i(T)} \to A[[\xi_i]]$. Set $B = \Gamma(C \setminus \cup_{i=1}^{n} p_i(T))$. Then using the $\eta_i$, there are maps $B \to A((\xi_i))$. One has that $\mathfrak{g} \otimes_k B$ maps to $\hat{\mathfrak{g}}_i \otimes_k A$, for each $i$, and moreover $H_{\vec{\lambda}} \otimes_k A$ is a representation of $\mathfrak{g} \otimes_k B$ via restriction and acting diagonally (as before). Define the sections of the sheaf of coinvariants $V_C(\vec{\lambda}, \vec{p})$ over $T$ to be the quotient $V_C(\vec{\lambda}, \vec{p}) = H_{\vec{\lambda}} \otimes_k A / (\mathfrak{g} \otimes_k B) \cdot H_{\vec{\lambda}} \otimes_k A$. If $T$ is not affine, then to define $V_C(\vec{\lambda}, \vec{p})$, one takes an open affine covering and extends by the sheaf property.

Furthermore, in this description, the open set $C \setminus \cup_{i=1}^{n} p_i(T)$ has been implicitly assumed to be affine. But this premise can be removed using a descent argument: See [Fak12, Prop 2.1], and the discussion following.

## 2.6 Appendix: Beauville's quotient construction

I mentioned Beauville's quotient construction a couple of times, and so I've included this sketch. The notes here closely follow [TUY89, Prop 2.2.6], [Bea96, Prop 2.3, Cor 2.4], and [BGM13]. We first state a general result, Theorem 2.6.1, which refers to the following, mentioned earlier in the notes, slightly differently.

**Notation**

Let $C$ be a possibly nodal curve, $p_1, p_2, \dots, p_s \in C$ be $s$ smooth points of $C$, $U = C\backslash\{p_1,\dots,p_s\}$ and let $\xi_i$ be a local parameter of $C$ near $p_i$. Then for $\hat{\mathfrak{g}}(U) = \mathfrak{g}\otimes\mathcal{O}_C(U)$, one can show the following is an embedding of Lie algebras:

$$\hat{\mathfrak{g}}(U) \hookrightarrow \bigotimes_{i=1}^{s} \left(\mathfrak{g}\otimes \mathrm{k}((\xi_i))\right) \oplus \mathrm{k}\, c = \hat{\mathfrak{g}}_s, \quad (\mathrm{X}\otimes f) \mapsto (\mathrm{X}\otimes f_{p_1}(\xi_1),\dots,\mathrm{X}\otimes f_{p_n}(\xi_n),0).$$

Given weights $\lambda_1, \dots, \lambda_s \in \mathcal{P}_\ell(\mathfrak{g})$, we have the $(\mathfrak{g}\otimes\mathrm{k}((\xi_i))\oplus\mathrm{k}\,c)$-modules $\mathrm{H}_{\lambda_i}$. The image of $\hat{\mathfrak{g}}(U)$ acts on $\mathrm{H}_{\vec{\lambda}} = \mathrm{H}_{\lambda_1}\otimes\cdots\otimes\mathrm{H}_{\lambda_s}$:

$$\hat{\mathfrak{g}}(U)\times\mathrm{H}_{\vec{\lambda}} \to \mathrm{H}_{\vec{\lambda}}, \ \ ((\mathrm{X}\otimes f),(w_1\otimes\cdots\otimes w_s)) \mapsto \sum_{i=1}^{s} w_1\otimes\cdots w_{i-1}\otimes(\mathrm{X}\otimes f_{p_i})\cdot w_i\otimes\cdots\otimes w_s.$$

Now given any weight $\mu\in\mathcal{P}_\ell(\mathfrak{g})$, recall that the subspace of $\mathrm{H}_\mu$ annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic as a $\mathfrak{g}$-module to $\mathrm{V}_\mu$, and so $\mathrm{V}_\mu$ is identified with this subspace of $\mathrm{H}_\mu$. Given $t$ points $q_1, \dots, q_t \in U$, and weights, $\mu_1, \mu_2, \dots, \mu_t \in \mathcal{P}_\ell(\mathfrak{g})$ one can define an action of $\hat{\mathfrak{g}}(U)$ on $\mathrm{V}_{\vec{\mu}} = \mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_t}$ by evaluation:

$$\hat{\mathfrak{g}}(U)\times\mathrm{V}_{\vec{\mu}} \to \mathrm{V}_{\vec{\mu}}, \ \ ((\mathrm{X}\otimes f),(v_1\otimes\cdots\otimes v_t)) \mapsto \sum_{j=1}^{t} v_1\otimes\cdots\otimes v_{j-1}\otimes(\mathrm{X}\otimes f(q_j))\cdot v_j\otimes\cdots\otimes v_t.$$

**Theorem 2.6.1.** *With notation as above, the inclusions* $\mathrm{V}_{\mu_j} \hookrightarrow \mathrm{H}_{\mu_j}$ *induce an isomorphism*

$$[\mathrm{H}_{\vec{\lambda}}\otimes\mathrm{V}_{\vec{\mu}}]_{\hat{\mathfrak{g}}(U)} \xrightarrow{\sim} [\mathrm{H}_{\vec{\lambda}}\otimes\mathrm{H}_{\vec{\mu}}]_{\hat{\mathfrak{g}}(U\backslash\vec{q})} \cong \mathbb{V}(\mathfrak{g},\vec{\lambda}\cup\vec{\mu},\ell)|_{(C,\vec{p}\cup\vec{q})}.$$

## 2.6.1 The construction

**Corollary 2.6.2.** *Let $q\in C\backslash\vec{p}$. There is a canonical isomorphism*

$$\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(C,\vec{p})} \cong [\mathrm{H}_0\otimes\mathrm{V}_{\vec{\lambda}}]_{\hat{\mathfrak{g}}(C\backslash q)}.$$

*Proof.* (of Corollary 2.6.2) By Corollary 2.4.1 (to Theorem 2.6.1), and the definition of vector spaces of covacua:

$$\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(C,\vec{p})} = \mathbb{V}(\mathfrak{g},\{0\}\cup\vec{\lambda},\ell)|_{(C,\{q\}\cup\vec{p})} = [\mathcal{H}_0\otimes\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(C\backslash\{q\}\cup\vec{p}}.$$

Now also by Theorem 2.6.1,

$$[\mathcal{H}_0\otimes\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(C\backslash\{q\}\cup\vec{p}} = [\mathcal{H}_0\otimes\mathrm{V}_{\vec{\lambda}}]_{\mathfrak{g}(C\backslash\{q\}}.$$

□

We briefly outline the proof of Theorem 2.6.1 in three steps. The full proof is given in [Bea96, pages 7-8].

**The proof of Theorem 2.6.1**

*Proof.* (of Theorem 2.6.1) We work by induction: Put $q = q_t$, $\mu = \mu_t$, $U = C\backslash\vec{p}$, and $\mathcal{H} = \mathrm{H}_{\vec{\lambda}} \otimes \mathrm{V}_{\mu_1} \cdots \otimes \mathrm{V}_{\mu_{t-1}}$. It will be enough to show that the inclusion $\mathrm{V}_\mu \hookrightarrow \mathrm{H}_\mu$ induces an isomorphism

$$[\mathcal{H} \otimes \mathrm{V}_\mu]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H} \otimes \mathrm{H}_\mu]_{\mathfrak{g}(U\backslash q)}.$$

**Step One.**

Show that the inclusion of $\mathrm{V}_\mu \hookrightarrow \mathrm{H}_\mu$ is equivariant with respect to the action of $\mathfrak{g}(U)$ so that it induces a linear map

$$[\mathcal{H} \otimes \mathrm{V}_\mu]_{\mathfrak{g}(U)} \to [\mathcal{H} \otimes \mathrm{H}_\mu]_{\mathfrak{g}(U\backslash q)}.$$

**Step Two.**

Prove the result when we replace $\mathrm{H}_\mu$ by the Verma module $\mathrm{M}_\mu$:

**Claim 2.6.3.**

$$[\mathcal{H} \otimes \mathrm{V}_\mu]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H} \otimes \mathrm{M}_\mu]_{\mathfrak{g}(U\backslash q)}.$$

*Proof.* (Outline) Choose a local coordinate $z$ at $q$ so that $z^{-1} \in \mathcal{O}_C(U\backslash q)$, and write

$$\mathfrak{g}(U\backslash q) = \mathfrak{g} \otimes \mathcal{O}_C(U\backslash q) = \mathfrak{g} \otimes \Big(\sum_{n\geqslant 1} \mathrm{k}\, z^{-n}\Big) = \mathfrak{g} \otimes \mathcal{O}_C(U) \oplus \Big(\sum_{n\geqslant 1} \mathfrak{g} z^{-n}\Big) = \mathfrak{g}(U) \oplus \hat{\mathfrak{g}}_-,$$

where we identify the Lie algebra $\sum_{n\geqslant 1} \mathfrak{g} z^{-n}$ with its image $\hat{\mathfrak{g}}_-$ in $\hat{\mathfrak{g}}$. So one wants to show

$$[\mathcal{H} \otimes \mathrm{V}_\mu]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H} \otimes \mathrm{M}_\mu]_{\mathfrak{g}(U)\oplus\hat{\mathfrak{g}}_-}.$$

We first show that

$$[\mathcal{H} \otimes \mathrm{M}_\mu]_{\hat{\mathfrak{g}}_-} \cong \mathcal{H} \otimes \mathrm{V}_{\vec{\lambda}}.$$

After doing so, taking coinvariants under $\mathfrak{g}$ will give the result.

By definition, $[\mathcal{H} \otimes \mathrm{M}_\mu]_{\hat{\mathfrak{g}}_-}$ is the same as the tensor product $\mathcal{H} \otimes_{U(\hat{\mathfrak{g}}_-)} \mathrm{M}_\mu$. Now by definition of $\mathrm{M}_\mu$,

$$\mathcal{H} \otimes_{U(\hat{\mathfrak{g}}_-)} \mathrm{M}_\mu \cong \mathcal{H} \otimes_{U(\hat{\mathfrak{g}}_-)} U(\hat{\mathfrak{g}}_-) \otimes_{\mathrm{k}} \mathrm{V}_{\vec{\lambda}} \cong \mathcal{H} \otimes_{\mathrm{k}} \mathrm{V}_{\vec{\lambda}}.$$

□

**Step Three.**

For $I_\mu$ such that $\mathrm{H}_\mu = \mathrm{M}_\mu / I_\mu$, one has the exact sequence:

$$\mathcal{H} \otimes I_\mu \to [\mathcal{H} \otimes \mathrm{M}_\mu]_{\mathfrak{g}(U\backslash q)} \to [\mathcal{H} \otimes \mathrm{H}_\mu]_{\mathfrak{g}(U\backslash q)} \to 0.$$

**Claim 2.6.4.** *The image of $\mathcal{H} \otimes I_\mu$ in $[\mathcal{H} \otimes \mathrm{M}_\mu]_{\mathfrak{g}(U\backslash q)}$ is zero.*

*Proof.* (Outline) Using that by definition, $[\mathcal{H}\otimes\mathrm{M}_\mu]_{\mathfrak{g}(U\backslash q)}$ is the same as $\mathcal{H}\otimes_{U(\mathfrak{g}(U\backslash q))}\mathrm{M}_\mu$, and that as a $U(\hat{\mathfrak{g}})$-module, $I_\mu$ is generated by the element

$$(X_\theta \otimes z^{-1})^{\ell-(\theta,\mu)+1} \otimes v_\mu,$$

where $v_\mu$ is the highest weight vector associated to $\mu$ and this vector is annihilated by $\hat{\mathfrak{g}}_+$, It is enough to show that $h \otimes ((X_\theta \otimes z^{-1})^{\ell-(\theta,\mu)+1} \cdot v_\mu) = 0$ for all $h \in \mathcal{H}$. This is done in [Bea96].

□

# Lecture 3

# Geometric interpretations of conformal blocks

## 3.1 Introduction

In the this lecture I will discuss algebro-geometric descriptions of the dual spaces to the fibers of the bundles $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ at points $(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n}$. We'll consider what we know when we take our points from the interior of the moduli space, when $C$ is a smooth curve, as well as what we know about such interpretations when $C$ has singularities, an open problem since the early 1990's.

I have selected this topic as it illustrates a theme that I would like to put forward: We have been able to learn about vector spaces of conformal blocks by studying the vector bundles they form on the moduli space, rather than as vector space alone.

As I'll explain today, there are points on $\overline{\mathrm{M}}_{g,n}$ for which no such geometric interpretation for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^{*}_{(C,\vec{p})}$ exists, whereas there are other points at which everything works out as it does for smooth curves. One way to approach a solution is to understand finite generation of the algebra of conformal blocks, which I will introduce.

## 3.2 Finite generation of the section ring of the determinant bundle

For G be a simple, simply connected, complex linear algebraic group, $C$ a stable curve of arithmetic genus $g \geqslant 2$, $\mathrm{Bun}_{\mathrm{G}}(C)$ is the smooth algebraic stack whose fiber over a scheme $T$ is the groupoid of principal G-bundles on $C \times T$ (Def 3.7.1). For completion, principal G- Bundles are defined carefully at the end of the lecture. To any representation $\mathrm{G} \to GL(V)$, there corresponds a distinguished line bundle on $\mathrm{Bun}_{\mathrm{G}}(C)$, the determinant of cohomology line bundle $\mathcal{D} = \mathcal{D}(V)$, described next.

To define $\mathcal{D}(V)$, we use the following:

**Definition 3.2.1.** *For any vector bundle $\mathcal{E}$ on a curve $C$, the determinant of cohomology of $\mathcal{E}$ on $C$ is the one dimensional vector space given by*

$$\mathcal{D}(C,\mathcal{E}) = \left(\Lambda^{max}\,\mathrm{H}^0\left(C,\mathcal{E}\right)\right)^* \otimes \left(\Lambda^{max}\,\mathrm{H}^1\left(C,\mathcal{E}\right)\right). \tag{3.1}$$

Following [LS97], we define the determinant of cohomology line bundle as follows.

**Definition 3.2.2.** *Let $\rho : \mathrm{G} \to \mathrm{Gl}(V)$ be a representation of G. If $E$ is a family of G-bundles on $C$ parameterized by a scheme $T$, then given a point $t \in T$, one has that $E_t$ is a G-bundle on $C$, and one can form a vector bundle $\mathcal{E}_t(V)$ on $C$ by taking the contracted product $\mathcal{E}_t(V) = E_t \times_{\mathrm{G}} V$. The determinant of cohomology line bundle $\mathcal{D}_E(V)$ is the line bundle on $T$ whose fiber over a point $t \in T$ is the line $\mathcal{D}(C,\mathcal{E}_t(V))$, described in Def 3.2.1.*

**Theorem 3.2.3.** *For $\mathrm{G} = \mathrm{SL}(r)$, for the standard representation $\mathrm{SL}(r) \to \mathrm{Gl}(V)$, setting $\mathcal{D} = \mathcal{D}(V)$,*

$$\mathcal{A}^C_\bullet = \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), \mathcal{D}^{\otimes m})$$

*is finitely generated.*

Theorem 3.2.3 was proved in case of smooth curves in [BL94], and [Fal94], and for stable curves with singularities in [BG16].

## 3.3 Geometric interpretations at smooth curves

To understand just what Theorem 3.2.3 has to do with conformal blocks, we consider the following results.

**Theorem 3.3.1.** $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|^*_{(C;\vec{p})} \cong \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathrm{H}^0(\mathrm{Bun}_{\mathrm{G}}(C), \mathcal{D}^{\otimes \ell m})$.

Theorem 3.3.1 was proved for smooth curves in [BL94], and for $C$ stable with singularities in [BF15]. In fact, this statement can be stated in full generality as follows:

**Theorem 3.3.2.** $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^*_{(C;\vec{p})} \cong \mathrm{H}^0(\mathrm{Parbun}_{\mathrm{G}}(C,\vec{p}), \mathcal{L}_{\mathrm{G}}(C,\vec{p},\vec{\lambda}))$.

The moduli stack $\mathrm{Parbun}_{\mathrm{G}}(C,\vec{p})$ maps to $\mathrm{Bun}_{\mathrm{G}}(C)$ and the line bundle $\mathcal{L}_{\mathrm{G}}(C,\vec{p},\vec{\lambda})$ on $\mathrm{Parbun}_{\mathrm{G}}(C,\vec{p})$ is constructed from $\mathcal{D}(\mathcal{V})$ on $\mathrm{Bun}_{\mathrm{G}}(C)$. Theorem 3.3.2 was proved for smooth curves by Laszlo and Sorger [LS97]. The result holds for families of singular stable curves by [BF15]. A simple alternative proof is given for singular stable curves in [BG16] (not families), that is inductive and uses factorization.

If $C$ is smooth, even more is true: stated in the case we are using now

**Theorem 3.3.3.** $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathrm{H}^0(\mathrm{Bun}_{\mathrm{G}}(C), \mathcal{D}^{\otimes \ell m}) \cong \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathrm{H}^0(X, A^{\otimes \ell m})$,
*where* $(X, A) = (\mathrm{M}_C(r), \theta)$ *is the projective polarized pair:*

- $X = \mathrm{M}_C(r)$ *is the moduli space parametrizing semi-stable vector bundles on $C$ of rank $r$ with trivializable determinant; and*
- $A = \theta = \{\mathcal{E} \in \mathrm{M}_C(r) \mid \mathcal{E} \otimes \mathcal{L}$ *has a nonzero section*$\}$, *for $\mathcal{L}$ a fixed line bundle on $C$ of rank $g-1$.*

*Putting Theorems 3.2.3, 3.3.1, and 3.3.3 together, we say that for a point $[C] \in \mathrm{M}_g$, corresponding to a smooth curve $C$, there is a projective polarized pair $(\mathrm{M}_C(r), \theta)$ such that*

$$\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|^*_{[C]} \cong \mathrm{H}^0(\mathrm{M}_C(r), \theta^{\ell m}),$$

*and so*

$$\mathrm{Proj}(\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|^*_{[C]}) \cong \mathrm{M}_C(r).$$

*In other words, there are geometric interpretations for conformal blocks at smooth curves. The same is true for conformal blocks at smooth pointed curves.*

**Example 3.3.4.** *For $[C] \in \mathrm{M}_2$, one has, that*

$$\mathbb{V}(\mathfrak{sl}_2, 1)|^*_C \cong \mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}(2)}(C), \mathcal{D}(V)) \cong \mathrm{H}^0(\mathrm{M}_C(2), \theta) \cong \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1)),$$

*where the third isomorphism was proved in a 1960's Annals paper by Narasimhan and Ramanan. More generally, we write*

$$\bigoplus_m \mathbb{V}(\mathfrak{sl}_2, m)|_C^* \cong \bigoplus_m \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(m)).$$

## 3.4 Geometric interpretations at stable curves

*We consider whether such geometric interpretations for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ exist at points $(C; \vec{p}) \in \overline{\mathrm{M}}_{g,n}$, where $C$ has singularities. We state this problem in the simplest case:*

**Question 3.4.1.** *Given a point $[C] \in \overline{\mathrm{M}}_g$, corresponding to a curve $C$ with singularities, is there is a projective polarized pair $(X, A)$ such that*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^* \cong \mathrm{H}^0(X, A^{\otimes m}),$$

*and so*

$$\mathrm{Proj}(\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^*) \cong X?$$

*We showed in [BGK16] that for this question, and the analogous more general question for conformal blocks on pointed curves, while sometimes yes, the answer is no, not necessarily!*

### 3.4.1 Negative results

**Example 3.4.2.** *[BGK16] Let $C$ be a stable curve of genus 2 with a separating node. There is no polarized pair $(X, A)$ such that*

$$\bigoplus_{m \in \mathbb{Z}} \mathbb{V}(\mathfrak{sl}_2, m)|_{[C]}^* \cong \bigoplus_{m \in \mathbb{Z}} H^0(X, A^m).$$

*To show this we prove that if $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, 1)$ has geometric interpretations at such a curve $C$, then*

$$c_1(\mathbb{V}(\mathfrak{sl}_2, m)) = \binom{m+3}{4} c_1(\mathbb{V}(\mathfrak{sl}_2, 1)) \tag{3.2}$$

*which one can show fails by intersecting with* F*-curves. There are two types of* F*-curves on $\overline{\mathrm{M}}_2$. The first is the image of a clutching map from $\overline{\mathrm{M}}_{0,4}$ for which points are identified in pairs. The second is the image of a map from $\overline{\mathrm{M}}_{1,1}$ given by attaching a point $(E, p) \in \mathrm{M}_{1,1}$,* *gluing the curves at the marked points. One obtains a contradiction when we intersect with either type of* F*-curve, even just at* $m = 2$*.*

*I'll talk about the proof of this and more general related results in my fourth lecture.*

**Example 3.4.3.** *[BGK16] For* $(C, \vec{p}) \in \overline{\mathrm{M}}_{2,n}$*, for* $n = 2k > 0$*, such that* $C$ *has a single separating node, then here is no polarized pair* $(X, A)$ *such that*

$$\bigoplus_{m \in \mathbb{Z}} \mathbb{V}(\mathfrak{sl}_2, \omega_1^n, m)|_{[C]}^{*} \cong \bigoplus_{m \in \mathbb{Z}} H^0(X, A^m).$$

*So that* $\mathbb{V}(\mathfrak{sl}_2, \omega_1^n, 1)$ *does not have geometric interpretations at such points* $(C, \vec{p}) \in \overline{\mathrm{M}}_{2,n}$*.*

*We do know that sometimes there are geometric interpretations. Here are two types of results along those lines:*

### 3.4.2 Positive result for positive genus

**Theorem 3.4.4.** *[BG16]Given* $[C] \in \overline{\mathcal{M}}_g$*, and a positive integer* $r$*, there exists a projective polarized pair* $(\mathcal{X}_C(r, \ell), \mathcal{L}_C(r, \ell))$*, and a positive integer* $\ell$ *such that*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, m\ell)|_{[C]}^{*} \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathrm{H}^0(\mathcal{X}_C(r, \ell), \mathcal{L}_C(r, \ell)^{\otimes m}). \tag{3.3}$$

*We can be more precise about* $\ell$ *in some cases:*

1. *For general* $r$ *if* $C$ *has only nonseparating nodes,* $\ell \geqslant 1$*;*
2. *For* $r = 2$*,* $\ell$ *divisible by* $2$*;*
3. *For general* $r$*, and* $C$ *with separating nodes, we know such an* $\ell$ *exists.*

**Example 3.4.5.** *So by Theorem 3.4.4, the bundle* $\mathbb{V}(\mathfrak{sl}_2, 1)$ *has geometric interpretations at a point* $[C] \in \overline{\mathrm{M}}_2$ *with only nonseparating nodes, even though it does not have if* $C$ *has a separating node, while* $\mathbb{V}(\mathfrak{sl}_2, 2)$ *has geometric interpretations at all points* $[C] \in \overline{\mathrm{M}}_2$*.*

*To prove Theorem 3.4.4, we use Theorems 3.2.3, and 3.3.1, together with the stratification of* $\overline{\mathrm{M}}_g$ *to prove that*

$$\mathcal{A}_\bullet = \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)^{*},$$

*is finitely generated.*

*The sheaves of conformal blocks* $\mathbb{V}(\mathfrak{sl}_r, \ell m)^*$ *are locally free of finite rank. This sum forms the so-called algebra of conformal blocks, mentioned in Falting's work, and studied by Chris*

*Manon mainly for* $\mathrm{SL}(2)$ *and* $\mathrm{SL}(3)$. *In these cases, Manon shows the algebra is finitely generated. Manon also takes* $\mathrm{Proj}(\mathcal{A}_\bullet)$ *in more general circumstances, without knowing or checking finite generation.*

*For* $\mathcal{A}_\bullet$ *to be finitely generated, it means that the algebra is generated over* $\mathcal{A}_0 \cong \mathcal{O}_{\overline{\mathrm{M}}_g}$ *by finitely many elements* $\{f_{d_i}\}_{i=1}^n$, *with* $f_{d_i} \in \mathcal{A}_{d_i} = \mathbb{V}(\mathfrak{sl}_r, d_i)^*$.

*For* $d = \Pi_{i=1}^n d_i$, *we let* $\mathcal{S}_\bullet = \bigoplus_m \mathcal{S}_m$, *where* $\mathcal{S}_m = \mathcal{A}_{dm}$, *be the d-th Veronese subring of* $\mathcal{A}_\bullet$. *Then* $\mathcal{S}_\bullet$ *is generated in degree* 1 *over* $\mathcal{S}_0$, *and*

$$\mathcal{X} := \mathrm{Proj}(\mathcal{A}_\bullet) \cong \mathrm{Proj}(\mathcal{S}_\bullet) \xrightarrow{p} \overline{\mathrm{M}}_g$$

*is a flat family.*

*Moreover, by definition, for* $k >> 0$,

$$\mathbb{V}(\mathfrak{sl}_r, kd)^* = \mathcal{S}_k \longrightarrow p_*\mathcal{O}_X(k),$$

*are isomorphisms. Since taking fibers commutes with taking* Proj,

$$p^{-1}([C]) = \mathcal{X}_C \cong \mathrm{Proj}(\bigoplus_m \mathbb{V}(\mathfrak{sl}_r, \ell m)|_C^*) = \mathrm{Proj}(\mathcal{A}_\bullet^{C,\ell}),$$

*where* $\mathcal{A}_\bullet^{C,1} = \mathcal{A}_\bullet^C$.

*By definition of pushforward,*

$$\mathcal{S}_k|_{[C]} = \mathbb{V}(\mathfrak{sl}_r, kd)|_{[C}^* = (p|_{\mathcal{X}_C})_*(\mathcal{O}_{\mathcal{X}}(kd)|_{\mathcal{X}_C}) \cong H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(kd)).$$

*In other words, for* $\ell = kd$, *and* $k >> 0$, *there is a projective polarized pair* $(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(\ell))$ *such that* $\mathbb{V}(\mathfrak{sl}_r, \ell)|_{[C]}^* \cong H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(\ell))$, *and*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^* \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(\ell m)).$$

*So* $\mathbb{V}(\mathfrak{sl}_r, \ell)$ *has geometric interpretations at C if* $\ell = kd$, *and* $k >> 0$.

**Remark 3.4.6.** *The flat family* $\mathcal{X} := \mathrm{Proj}(\mathcal{A}_\bullet) \cong \mathrm{Proj}(\mathcal{S}_\bullet) \xrightarrow{p} \overline{\mathrm{M}}_g$ *is one way to complete the family* $\mathcal{X}^0 \xrightarrow{p} \mathrm{M}_g$ *whose fibers over points* $[C]$ *are the moduli spaces* $\mathrm{M}_C(r)$ *described earlier. There are other ways to complete this family and this problem is an old one with an interesting history.*

### 3.4.3 Positive result for bundles of rank one

**Theorem 3.4.7.** *[BGK16] Geometric interpretations hold at all points if $\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ has rank one.*

*More general results hold for bundles with restriction behavior that is similar to that for rank one bundles. We avoid stating these results here because they are involved.*

*While I don't know of any vector bundle of conformal blocks of rank one on $\overline{\mathrm{M}}_{g,n}$ for positive genus g, every bundle on $\overline{\mathrm{M}}_{0,n}$ of the form $\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, 1)$ has rank one, and by Theorem 3.4.7, all such bundles have geometric interpretations at all points of $\overline{\mathrm{M}}_{0,n}$.*

**Example 3.4.8.** *For contrast with Example 3.4.3, $\mathbb{V}(\mathfrak{sl}_2, \omega_1^{2k}, 1)$ has rank one on $\overline{\mathrm{M}}_{0,2k}$, and by Theorem 3.4.7, geometric interpretations at all points of $\overline{\mathrm{M}}_{0,2k}$, whereas by [BGK16] the same bundle on $\overline{\mathrm{M}}_{2,2k}$ will not have geometric interpretations at a point $(C, \vec{p})$ if C has a separating node.*

**Idea of proof of Theorem 3.4.7**

## 3.5 Idea behind the proof of the negative result

*In [BGK16] we prove the following:*

**Theorem 3.5.1.** *There are points $(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n}$ and vector bundles of conformal blocks $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ on $\overline{\mathrm{M}}_{g,n}$ for which there is no projective polarized pair for which*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)|^*_{(C,\vec{p})} \cong \bigoplus_{m \in \mathbb{Z}} H^0(X, A^m) \tag{3.4}$$

*holds.*

*To prove Theorem 3.5.1, we give obstructions to geometric interpretations for those bundles where geometric interpretations at smooth curves are known to be varieties of minimal degree. I'll explain what I mean next.*

*Given a projective polarized pair $(X, A)$, there is a quantity called the $\Delta$-invariant or $\Delta$-genus, which is defined to be*

$$\Delta(X, A) = \dim(X) + A^{\dim(X)} - h^0(X, A).$$

*Fujita (1990, Chapter 1 [**?**]) proved that $\Delta(X, A) \geqslant 0$, and if $\Delta(X, A) = 0$, the section ring of $A$, $\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} H^0(X, A^{\otimes m})$ is generated by its global sections $H^0(X, A)$, and so $A$ is very ample. In this case, when $A$ is very ample, it gives an embedding of $X$ into projective space*

$$X \hookrightarrow \mathrm{Proj}(B_\bullet) = \mathbb{P}^N, \;\; B_\bullet = \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} Sym^m(H^0(X, A)).$$

*The image of $X$ is a non-degenerate variety of degree*

$$A^{\dim X} = 1 + codim(X).$$

*A non-degenerate variety $X \hookrightarrow \mathbb{P}^N$ is of minimal degree if $\deg(X) = 1 + codim(X)$.*

*So if $(X, A)$ is a projective polarized pair with $\Delta(X, A) = 0$, then the image of the variety $X$ embedded by $A$ is a variety of minimal degree.*

*Varieties of minimal degree are classified. For instance $(X, A) \cong (\mathbb{P}^d, \mathcal{O}(1))$ if and only if $A^d = 1$.*

*What is crucial to our line of reasoning is that the $\Delta$-invariant is upper semi-continuous: If $\mathbb{V}$ is a vector bundle of conformal blocks on $\overline{\mathrm{M}}_{g,n}$ that has geometric interpretation at some point $(C, \vec{p}) \in \mathrm{M}_{g,n}$ such that the corresponding projective polarized pair has $\Delta$-invariant zero, then if it has geometric interpretations at any other points, those corresponding pairs will also have $\Delta$-invariant zero.*

*We use this to prove the following result (paraphrased):*

**Theorem 3.5.2.** *Suppose that $\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)$ has $\Delta$-invariant zero rank scaling, and geometric interpretations exist for $\mathbb{V}$ at all points, then for all $m$, $c_1(\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell))$ can be expressed in terms of $c_1(\mathbb{V}(\mathfrak{g}, k\vec{\lambda}, k\ell))$, for $k < m$.*

*There is an explicit statement for Theorem 3.5.2, which is rather long and technical. In Example 3.5.3, Theorem 3.5.2 is stated for the stronger case of projective rank scaling, where there is an if and only if result.*

**Example 3.5.3.** $\mathrm{Rank}(\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)) = \binom{m+d}{d}$, *and geometric interpretations exist for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ at all points $(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n}$, iff $c_1(\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)) = \binom{m+d}{d+1} c_1\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$. In particular, if $d = 0$, so that the rank is one, we know by [GG12] for $\mathfrak{sl}_r$ and $\ell = 1$, and by [BGK16] for the general case,*

$$c_1\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell) = mc_1\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell).$$

*Therefore for rank one bundles, geometric interpretations exist at all points.*

*We can outline the proof of Theorem 3.5.2 in two steps:*

1. *Suppose that for every point $x \in \overline{\mathrm{M}}_{g,n}$, there is a projective polarized pair $(X_x, A_x)$ of $\Delta$-invariant zero so that there is a canonical embedding as described above. One can then take the canonical resolution of the ideal sheaf $\mathcal{I}_{X_x}$ for $X_x$.*
2. *By "glueing" the resolutions, we show there is an exact sequence*

   $$0 \to \mathcal{W}_D \otimes Sym^{m-D}(\mathbb{V}) \to \cdots \to \mathcal{W} \otimes Sym^{m-1}(\mathbb{V}) \to Sym^m(\mathbb{V}) \to \mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)^* \to 0, \tag{3.5}$$

   *where the $\mathcal{W}_i$ are vector bundles on $\overline{\mathrm{M}}_{g,n}$.*

## 3.6 Idea of proof of Theorem 3.2.3

*The proof of Theorem 3.2.3 can be outlined in four steps:*

1. *Define projective polarized pairs $(X(\vec{a}), \mathcal{L}(\mathcal{G}))$, where $X(\vec{a})$ is a compactification of a moduli space of $\vec{a}$-semistable vector bundles of rank r on C with trivializable determinant. The compactification is obtained as a GIT quotient of torsion free sheaves. The semi-stability condition is new; a generalization based on Seshadri and Simpson.*
2. *Show there are injections $H^0(X(\vec{a}), \mathcal{L}(\mathcal{G})) \hookrightarrow H^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), D(V)^m)$ giving rise to a map*

   $$F : \bigoplus_{(\vec{a}, \mathcal{G})} H^0(X(\vec{a}), \mathcal{L}(G)) \longrightarrow \bigoplus_{m \in \mathbb{Z}} H^0(\mathrm{Bun}_{\mathrm{SL}(r)}(C), D(V)^m).$$
3. *Using conformal blocks, show that F is surjective. For this we use Theorem 3.3.2 and Factorization. This involves a technical argument showing that certain sections extend across poles.*
4. *Show that the part of left hand side necessary for the surjection of F is finitely generated. This is achieved by noticing that the varieties $X(\vec{a})$ which are involved are all Geometric Invariant Theory (GIT) quotients of the same (master) space, and so one can use a variation of GIT argument to get the claim.*

## 3.7 Appendix: Definition of Principal G-bundles

**Definition 3.7.1.** *Let* G *be an algebraic group,* $X$ *a variety, and* $T$ *a Grothendieck topology. A principal* G*-bundle on* $X$ *with respect to* $T$*, is a morphism* $\pi : P \to X$ *together with an action* $P \times G \xrightarrow{a} P$ *such that the following properties hold:*

1. *The diagrams*

$$
\begin{array}{ccc}
P \times G & \xrightarrow{a} & P \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
P & \xrightarrow[\pi]{} & X
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
P \times G \times G & \xrightarrow{id \times \mu} & P \times G \\
\downarrow{\scriptstyle a \times id} & & \downarrow{\scriptstyle a} \\
P \times G & \xrightarrow[a]{} & P,
\end{array}
$$

   *commute, where* $\mu : G \times G \to G$ *denotes the multiplication operation on* $G$*.*

2. *There exists a covering* $\{\cup_{j \in J} U_j \to X\}$ *of* $X$ *in the* $T$ *topology, for which for each* $j \in J$ *there are* $G$*-space isomorphisms* $\psi_j : P|_{U_j} \xrightarrow{\cong} U_j \times G$*, meaning that the following two diagrams*

$$
\begin{array}{ccc}
P|_{U_j} & \xrightarrow{\psi_j} & U_j \times G \\
\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_1} & \\
U_j & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
P|_{U_j} \times G & \xrightarrow{a} & P|_{U_j} \\
\downarrow{\scriptstyle \psi_j \times id} & & \downarrow{\scriptstyle \psi_j} \\
U_j \times G \times G & \xrightarrow[id \times \mu]{} & U_j \times G
\end{array}
$$

   *commute.*

**Remark 3.7.2.** *If* $X$ *is defined over a field of char* 0*, then the fppf and etale topologies are the same. If* G *is simply connected and* $X$ *is a curve, as in our situation, then this is the same as working with the Zariski topology.*

*We now return to using the notation of* G*-bundles on* $C$ *where* $C$ *is a stable curve.*

**Definition 3.7.3.** *Let* $\rho : \mathrm{G} \to \mathrm{Gl}(V)$ *be a representation of* G*. If* $E$ *is a family of* G*-bundles on* $C$ *parameterized by a scheme* $T$*, then given a point* $t \in T$*, one has that* $E_t$ *is a* G*-bundle on* $C$*, and one can form a vector bundle* $\mathcal{E}_t(V)$ *on* $C$ *by taking the contracted product* $\mathcal{E}_t(V) = E_t \times_G V$*. The determinant of cohomology line bundle* $\mathcal{D}_E(V)$ *is the line bundle on* $T$ *whose fiber over a point* $t \in T$ *is the line* $\mathcal{D}(C, \mathcal{E}_t(V))$*, described in Def 3.2.1.*

**Definition 3.7.4.** *Let* G *be an algebraic group,* $\pi : E \to C$ *a principal* G*-bundle on* $C$*, and* $\rho : G \to \mathrm{GL}(V)$ *any representation. The contracted product* $\mathcal{E} = E \times_G V = (E \times V)/\sim$*, where* $(p \cdot g, v) \sim (p, \rho(g) \cdot v)$*, is a vector bundle on* $C$*, with fibers isomorphic to* $V$*: Given* $x \in C$*:*

$$\mathcal{E}|_x = E_x \times_G V \cong G \times_G V \cong V.$$

*I referred to the following fact:*

**Lemma 3.7.5.** *Let $G$ be any semisimple group. Given a principal* G*-bundle $\mathcal{E}$, and any representation $\rho : G \to \mathrm{GL}(V)$, by the contracted product $E = \mathcal{E} \times_G V$, has trivial determinant.*

*Proof.* To see that $\det(E)$ is trivial, we note that since $G$ is semisimple, $[G, G] = G$, and so the image $\rho(\mathrm{G})$ is contained in the kernel of the determinant map which is $\mathrm{SL}(V)$. In particular, $E$ has transition functions given by matrices with trivial determinant. These are the transition functions of the line bundle $\det(E)$, and so $\det(E)$ is necessarily trivial. □

# Lecture 4

# First Chern classes: vanishing, identities, applications, and open questions

## 4.1 Introduction

*For $g = 0$, and $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{g})^n$, the vector bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ is a quotient of the constant bundle:*

$$\mathbb{A}(\mathfrak{g}, \vec{\lambda}) = \mathrm{A}(\mathfrak{g}, \vec{\lambda}) \times \overline{\mathrm{M}}_{0,n} \twoheadrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell), \text{ where } \mathrm{A}(\mathfrak{g}, \vec{\lambda}) = [\mathrm{V}_{\lambda_1} \otimes \cdots \otimes \mathrm{V}_{\lambda_n}]_{\mathfrak{g}},$$

*is the vector space of coinvariants. So for every point $(C, \vec{p}) \in \overline{\mathrm{M}}_{0,n}$, there is a surjective map of vector spaces*

$$\mathrm{A}(\mathfrak{g}, \vec{\lambda}) \twoheadrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}.$$

*In other words, for*

$$a = \mathrm{rk}(\mathbb{A}(\mathfrak{g}, \vec{\lambda})), \text{ and } r = \mathrm{rk}(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)),$$

*there is a composition of morphisms*

$$\overline{\mathrm{M}}_{0,n} \xrightarrow{\phi} \mathrm{Gr}^{\mathrm{quo}}(\mathrm{A}(\mathfrak{g}, \vec{\lambda}), r) \xrightarrow{p} \mathbb{P} = \mathbb{P}^{\binom{a}{r}-1}, (C, \vec{p}) \mapsto [\Lambda^a \mathrm{A}(\mathfrak{g}, \vec{\lambda}) \twoheadrightarrow \Lambda^r(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}],$$

*and the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}$ is the conformal blocks divisor:*

$$(p \circ \phi)^* \mathcal{O}_{\mathbb{P}}(1) = c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)) = \mathbb{D}(\mathfrak{g}, \vec{\lambda}, \ell).$$

*If $\mathbb{A}(\mathfrak{g}, \vec{\lambda}) \cong \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$, then the two bundles have the same rank: $a = r$, and for $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$, the map $\phi_{c_1(\mathbb{V})} = (p \circ \phi)$ will contract everything in $\overline{\mathrm{M}}_{0,n}$ onto the point $\mathbb{P} =$*

$\mathbb{P}^{\binom{a}{r}-1} = \mathbb{P}^0$. *As will be explained, this always happens if* $\ell >> 0$. *In case* $\mathfrak{g} = \mathfrak{sl}_r$, *specifically for* $\ell$ *above the so-called* critical level *(Definition 4.2.1, and more generally for* $\ell$ *above the so-called* theta level *(Definition 4.4.1). As will be clear, the critical and theta levels are the same for* $\mathfrak{sl}_2$, *but generally measure different aspects of the bundles. There are plenty of divisors that are unexpectedly trivial, and so there may be some other level besides the critical and theta levels that control vanishing.*

*This kind of vanishing is important for understanding the placement of the divisors in the nef cone, which reflects the curves contracted by their associated morphisms. It also gives us some information about how much of the nef cone the divisors may generate.*

*For instance, applications of vanishing above the critical level include criteria that guarantee the divisors* $c_1(\mathbb{V})$ *intersect certain curves in degree zero: Said otherwise, the associated maps* $\phi_{c_1(\mathbb{V})}$ *contract those curves. In particular, we can use vanishing above the critical level to show that maps* $\phi_{c_1(\mathbb{V})}$ *factor through contractions from* $\overline{\mathrm{M}}_{0,n}$ *to Hassett spaces* $\overline{\mathrm{M}}_{0,\mathcal{A}}$, *where the weight data* $\mathcal{A}$ *is determined by the triple* $(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$.

*It seems the upshot is that conformal blocks divisors are quite often extremal in the nef cone, and the number of curves they contract increases as the level increases with respect to the pair* $(\mathfrak{g}, \vec{\lambda})$.

*It is an interesting question to determine just what divisors are nontrivial, and we discuss this today too.*

## 4.2 Vanishing above the critical level

### 4.2.1 Definition

**Definition 4.2.1.** *If* $r+1$ *divides* $\sum_{i=1}^n |\lambda_i|$, *we refer to*

$$cl(\mathfrak{sl}_{r+1}, \vec{\lambda}) = -1 + \frac{\sum_{i=1}^n |\lambda_i|}{r+1},$$

*as the critical level for the pair* $(\mathfrak{sl}_{r+1}, \vec{\lambda})$. *If* $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, *and if* $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})^n$, *then* $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ *is called a critical level bundle, and* $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ *is called a critical level divisor.*

### 4.2.2 Vanishing and identities

*Note that if* $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$*, then* $r = cl(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T)$*, where* $\vec{\lambda}^T = (\lambda_1^T, \ldots, \lambda_n^T)$*. Here* $\lambda_i^T$ *is the weight associated to the transpose of the Young diagram associated to the weight* $\lambda_i$*. In particular,* $|\lambda_i| = |\lambda_i^T|$*, and so*

$$\sum_{i=1}^{n} |\lambda_i| = (r+1)(\ell+1) = (\ell+1)(r+1) = \sum_{i=1}^{n} |\lambda_i^T|.$$

*In particular, critical level bundles come in pairs, and as we shall prove:*

**Theorem 4.2.2.** *[BGM15b] If* $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$*, then*

1. $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell + c)) = 0$*, for* $c \geqslant 1$*; and*
2. $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T, r))$.

*After giving some examples and applications, we will prove part* $(1)$ *of Theorem 4.2.2.*

### 4.2.3 Examples

1. *The bundle* $\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)$ *is at the critical level for* $n = (r+1)(\ell+1)$*. In [BGM15b] we showed that the first Chern classes are all nonzero, and by Theorem 4.2.2, for* $n = (r+1)(\ell+1)$*,*

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r)); \text{ and}$$

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell + c)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r + c)) = 0 \text{ for all } c \geqslant 1.$$

2. *The bundle* $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, (2\omega_1 + \omega_3)^3\}, 3)$ *is at the critical level, and its first Chern class is self dual.*

## 4.3 Applications

*The main applications of vanishing above the critical level are extremality tests, and criteria for showing that maps given by conformal blocks divisors factor through contraction maps to Hassett spaces.*

**Extremality test**

**Proposition 4.3.1.** *Let $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})^n$, and suppose that $N_1, N_2, N_3, N_4$ is a partition of $[n] = \{1, \dots, n\}$ into four nonempty subsets ordered so that if $\lambda(N_i) = \sum_{j \in N_i} |\lambda_j|$, then $\lambda(N_1) \leqslant \cdots \leqslant \lambda(N_4)$. If $\sum_{j \in \{1,2,3\}} \lambda(N_j) \leqslant r + \ell$, then*

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) \cdot F_{N_1,N_2,N_3} = 0,$$

*and in particular, $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell))$ is extremal in the nef cone.*

*Proof.* The intersection $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) \cdot F_{N_1,N_2,N_3}$ takes place in the boundary divisor

$$\Delta_{N_1 \cup N_2 \cup N_3} \cong \overline{\mathrm{M}}_{0,|N_1 \cup N_2 \cup N_3|+1} \times \overline{\mathrm{M}}_{0,|N_4|+1},$$

and in particular, in $\overline{\mathrm{M}}_{0,|N_1 \cup N_2 \cup N_3|+1}$. We can use factorization to examine the first Chern class of the bundle $\mathbb{V}$ at points $p \in \Delta_{N_1 \cup N_2 \cup N_3}$:

$$c_1(\mathbb{V})|_p \cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})} c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu, \ell)) c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_4) \cup \mu^*, \ell)).$$

We compute the critical level of $\mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu, \ell)$, which is

$$\begin{aligned} cl(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu) &= -1 + \frac{\sum_{j \in N_1 \cup N_2 \cup N_3} |\lambda_j| + |\mu|}{r+1} \\ &\leqslant -1 + \frac{r + \ell + r\ell}{r+1} < -1 + \frac{r + \ell + r\ell + 1}{r+1} = -1 + \frac{(r+1)(\ell+1)}{r+1} = \ell. \end{aligned} \tag{4.1}$$

In particular, $\mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu, \ell)$ is above the critical level, and so it has trivial first Chern class. □

**Criteria for mapping through Hassett Spaces**

**Definition 4.3.2.** *Fix a partition of $[n] = \{1, \dots, n\}$ into four nonempty sets $N_1$, $N_2$, $N_3$, $N_4 = [n] \backslash N_1 \cup N_2 \cup N_3$, and consider the morphism $\overline{\mathrm{M}}_{0,4} \longrightarrow \overline{\mathrm{M}}_{0,n}$, where $(C, (a_1, a_2, a_3, a_4)) \mapsto (X, (p_1, \dots, p_n))$, where $X$ is the nodal curve obtained as follows. If $|N_i| \geqslant 2$, then one glues a copy of $\mathbb{P}^1$ to the spine $(C, (a_1, a_2, a_3, a_4))$ by attaching a point $(\mathbb{P}^1, \{p_j : j \in N_i\} \cup \{\alpha_i\}) \in M_{0,|N_i|+1}$ to $a_i$ at $\alpha_i$. If $|N_i| = 1$, one does not glue any curve at the point $a_i$, but instead labels $a_i$ by $p_i$. We refer to any element of the numerical equivalence class of the image of this morphism the* F-Curve *$F(N_1, N_2, N_3)$ or by $F(N_1, N_2, N_3, N_4)$, depending on the context.*

### Background on Hassett spaces

*Consider an n-tuple* $\mathcal{A} = \{a_1, \ldots, a_n\}$, *with* $a_i \in \mathbb{Q}$, $0 < a_i \leqslant 1$, *such that* $\sum_i a_i > 2$. *In [Has03], Hassett introduced moduli spaces* $\overline{\mathrm{M}}_{0,\mathcal{A}}$, *parameterizing families of stable weighted pointed rational curves* $(C, (p_1, \ldots, p_n))$ *such that (1) C is nodal away from its marked points* $p_i$; *(2)* $\sum_{j\in J} a_i \leqslant 1$, *if the marked points* $\{p_j : j \in J\}$ *coincide; and (3) If* $C'$ *is an irreducible component of C then* $\sum_{p_i \in C'} a_i + $ *number of nodes on* $C' > 2$. *These Hassett spaces* $\overline{\mathrm{M}}_{0,\mathcal{A}}$ *receive birational morphisms* $\rho_{\mathcal{A}}$ *from* $\overline{\mathrm{M}}_{0,n}$ *that are characterized entirely by which F-Curves (see Def. 4.3.2) they contract.*

**Definition/Lemma 4.3.3.** *For any Hassett space* $\overline{\mathrm{M}}_{0,\mathcal{A}}$, *with* $\mathcal{A} = \{a_1, \ldots, a_n\}$, *there are birational morphisms* $\rho_{\mathcal{A}} : \overline{\mathrm{M}}_{0,n} \longrightarrow \overline{\mathrm{M}}_{0,\mathcal{A}}$, *contracting all* F*-curves* $F(N_1, N_2, N_3, N_4)$ *satisfying:* $\sum_{i\in N_1\cup N_2\cup N_3} a_i \leqslant 1$, *and no others, where without loss of generality, the leg* $N_4$ *carries the most weight.*

### Results on Hassett spaces

*The following theorem, proved in [BGM15b] generalizes [Fak12, Proposition 4.7], where* $\mathfrak{g} = \mathfrak{sl}_2$ *was considered.*

**Theorem 4.3.4.** *Let* $\mathbb{D} = \mathbb{D}_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell}$ *be such that:*

1. $0 < |\lambda_i| \leqslant \ell + r$ *for all* $i \in \{1, \ldots, n\}$;
2. $\sum_{i=1}^{n} |\lambda_i| > 2(r + \ell)$.

*Then the morphism* $\phi_{\mathbb{D}}$ *factors through* $\rho_{\mathcal{A}} : \overline{\mathrm{M}}_{0,n} \longrightarrow \overline{\mathrm{M}}_{0,\mathcal{A}}$, *where* $\mathcal{A} = \{a_1, \ldots, a_n\}$, $a_i = \frac{|\lambda_i|}{r+\ell}$.

*Proof.* For $\mathcal{A} = \{a_1, \ldots, a_n\}$, $a_i = \frac{|\lambda_i|}{r+\ell}$, as in the hypothesis, the condition $|\lambda_i| \leqslant \ell + r$, guarantees that $a_i \leqslant 1$ for all $i$, and $\sum_{i=1}^{n} |\lambda_i| > 2(r + \ell)$ guarantees that $\sum_{i=1}^{n} a_i > 2$.

By [Fak12, Lemma 4.6], we need to show that any F-curve $F(N_1, N_2, N_3, N_4)$ contracted by $\rho_{\mathcal{A}}$ is also contracted by $\phi_{\mathbb{D}}$. Suppose that $\rho_{\mathcal{A}}$ contracts the F-curve $F(N_1, N_2, N_3, N_4)$, so that in particular, by Definition/Lemma 4.3.3, $\sum_{i\in N_1\cup N_2\cup N_3} a_i \leqslant 1$. Then $\sum_{j\in\{1,2,3\}} \lambda(N_j) = (r + \ell) \sum_{j\in\{1,2,3\}} a_j \leqslant r + \ell$, and hence $\phi_{\mathbb{D}}$ contracts F-curve $F(N_1, N_2, N_3, N_4)$. ◻

### 4.3.1 Examples: GIT and images of conformal blocks maps

*While we have shown that many conformal blocks divisors give rise to maps that factor through Hassett spaces, their images are not in general isomorphic to Hassett spaces. As the following examples show, it does seem a likely possibility, that the type A conformal blocks divisors give maps to spaces that have modular interpretations and have constructions as GIT quotients. These particular examples have images that are birational to $\overline{\mathrm{M}}_{0,n}$, but in [BGM15a], we exhibit a divisor for which all the weights are nonzero and whose corresponding morphism has positive dimensional fibers.*

*If $\mathbb{D} = \mathbb{D}_{\mathfrak{sl}_{r+1},\vec{\lambda},1}$ is a nontrivial level one divisor, so that necessarily $\sum_i |\lambda_i| = (r+1)(d+1)$, for some $d \geqslant 1$, then the image of $\phi_{\mathbb{D}}$ is isomorphic to the generalized Veronese quotient $U_{d,n}/\!\!/_{(0,A)}\,\mathrm{SL}(d+1)$, where $a_i = |\lambda_i|/(1+r)$ [Gia13, GG12]. These spaces, which receive morphisms from $\overline{\mathrm{M}}_{0,\mathcal{A}}$, are birational to $\overline{\mathrm{M}}_{0,n}$ and have modular interpretations [Gia13, GJM13]. In [BGM15a], we prove that for all $\ell \geqslant 1$, the images of maps given by divisors $\mathbb{D}_{\mathfrak{sl}_{r+1},\ell\vec{\lambda},\ell}$ are also isomorphic to $U_{d,n}/\!\!/_{(0,\mathcal{A})}\,\mathrm{SL}(d+1)$.*

*In [BGM15a] it was shown that for all $\ell \geqslant 1$, and $r \geqslant 1$, the divisor $\mathbb{D} = \mathbb{D}_{\mathfrak{sl}_{r+1},\omega_1^n,\ell}$ is non-trivial. Because it is $S_n$-invariant, by [KM13, Gib09], it is big, and the corresponding morphism $\phi_{\mathbb{D}}$ is birational. By [BGM15a], for $\mathcal{A} = (\frac{1}{\ell+r},\ldots,\frac{1}{\ell+r})$, $\ell > 1$, and $r > 1$, the maps $\rho_{\mathcal{A}}$ and $\phi_{\mathbb{D}}$ contract the same F-curves. According to the F-conjecture, the divisors $\mathbb{D}$ and $\rho^*_{\mathcal{A}}(A)$, where $A$ is any ample divisor on $\overline{\mathrm{M}}_{0,\mathcal{A}}$ conjecturally lie on the same face of the nef cone of $\overline{\mathrm{M}}_{0,n}$. In particular, the (normalization of the) image of the morphism $\phi_{\mathbb{D}}$ should be isomorphic to $\overline{\mathrm{M}}_{0,\mathcal{A}}$. Moon has shown that $\overline{\mathrm{M}}_{0,\mathcal{A}}$ can be constructed as a GIT quotient of $\overline{\mathrm{M}}_{0,\mathcal{A}}(\mathbb{P}^1,1)$ by $\mathrm{SL}(2)$. The case $\ell = 1$, the image of $\phi_{\mathbb{D}}$ was shown in [Fak12] to be isomorphic to $(\mathbb{P}^1)^n/\!\!/_{\mathcal{A}}\,\mathrm{SL}(2)$, where $a_i = 1/(r+1)$. In case $r = 1$, the image of $\phi_{\mathbb{D}}$ was shown in [GJMS13] to be isomorphic to $U_{\ell,n}/\!\!/_{(\delta,\mathcal{A})}\,\mathrm{SL}(\ell+1)$, where $\delta = \frac{\ell-1}{\ell+1}$, $a_i = \frac{1}{\ell+1}$.*

### 4.3.2 Sketch of proof of vanishing above the critical level

*To prove Part (1) of Theorem 4.2.2, we use the cohomological version of Witten's Dictionary, which follows from [Wit95] and the twisting procedure of [Bel08], see Eq (3.10) from [Bel08].*

**Theorem 4.3.5.** *Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1},\vec{\lambda},\ell)$ be a vector bundle on $\overline{\mathcal{M}}_{g,n}$ such that $\sum_{i=1}^n |\lambda_i| = (r+1)(\ell+s)$ for some integer $s$.*

1. *If $s > 0$, then let $\lambda = \ell\omega_1$. The rank of $\mathbb{V}$ is the coefficient of $q^s\sigma_{\ell\omega_{r+1}}$ in the quantum*

*product*

$$\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \star \sigma_\lambda^s \in \mathrm{QH}^*(\mathrm{Gr}(r+1, r+1+\ell)).$$

2. *If $s \leqslant 0$, then the rank of $\mathbb{V}$ is the multiplicity of the class of a point $\sigma_{k\omega_{r+1}}$ in the product*

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \cdots \cdot \sigma_{\lambda_n} \in H^*(\mathrm{Gr}(r+1, r+1+k)),$$

*where $k = \ell + s$.*

*Examples of rank computations using Theorem 4.3.5 can be found in [BGM15b, BGM15a, Kaz16, Hob15] and [BGK16].*

*Proof.* Write $\tilde{\ell} = cl(\mathfrak{sl}_{r+1}, \vec{\lambda}) + 1$. We'll consider the following two cases:

1. $\vec{\lambda} \in \mathcal{P}_{\tilde{\ell}}(\mathfrak{sl}_{r+1})^n$ so that $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})$ makes sense, and there is a surjective map

$$\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda}) \twoheadrightarrow \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell}).$$

2. $\vec{\lambda} \notin \mathcal{P}_{\tilde{\ell}}(\mathfrak{sl}_{r+1})^n$.

In case $\vec{\lambda} \in \mathcal{P}_{\tilde{\ell}}(\mathfrak{sl}_{r+1})^n$, we know that by Beauville's quotient construction, as the level grows, the rank decreases:

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})) \leqslant \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell) \leqslant \mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda}).$$

So it is enough to show in this case that

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})) = \mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda}).$$

In the second case, we'll argue that $\mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda}) = 0$. Both follow from the Cohomological form of Witten's Dictionary, Theorem 4.3.5.

In the first case, since $\sum_{i=1}^n |\lambda_i| = (r+1)(\tilde{\ell})$, we have that $s = 0$ in Theorem 4.3.5, and so $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})) = \mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda})$, as claimed.

In the second case, we know that $|\lambda_i| \leqslant \ell r$ for all $i$ but that $|\lambda_i| > \tilde{\ell} r$ for some $i$. This means in particular that $\lambda_i^{(1)} > \tilde{\ell}$ for some $i$. We may reliable so that

$$k = \lambda_1^{(1)} \geqslant \cdots \lambda_n^{(1)}.$$

Since $\sum_{i=1}^n |\lambda_i| < (r+1)k$, we write $\sum_{i=1}^n |\lambda_i| = (r+1)(k-p)$, for some $p > 0$. Setting $\mu_1 = \mu_2 = \cdots = \mu_p = \omega_{r+1} \cong \omega_0$, by Propagation of Vacua:

$$\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda} \cup \vec{\mu}, \ell) \cong \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell),$$

and since $\sum_{i=1}^{n} |\lambda_i| + \sum_{j=1}^{p} |\mu_j| = (r+1)(k-p) + (r+1)p = (r+1)k$, we can compute the rank by computing the intersection

$$\sigma_{\lambda_1} \star \cdots \star \sigma_{\lambda_n} \star \sigma^{p}_{\omega_{r+1}} \in \mathrm{Gr}(r+1, r+1+k).$$

By a calculation, this is zero. □

## 4.4 The theta level

*The theta level (Def 4.4.1), comes from the interpretation of a vector space of conformal blocks as an explicit quotient [Bea96, Proposition 4.1] see also [?]), and holds in all types.*

### 4.4.1 Definition of the theta level

**Definition 4.4.1.** *[BGM15b] Given a pair $(\mathfrak{g}, \vec{\lambda})$, one refers to*

$$\theta(\mathfrak{g}, \vec{\lambda}) = -1 + \frac{1}{2}\sum_{i=1}^{n}(\lambda_i, \theta) \in \frac{1}{2}\mathbb{Z}$$

*as the theta level of the pair $(\mathfrak{g}, \vec{\lambda})$.*

**Remark 4.4.2.** *In Definition 4.4.1, as described in Lecture 2.1, $\theta$ is the highest root, and $(\ ,\ )$ is the normalized Killing form.*

### 4.4.2 Vanishing above the theta level

*Using [Bea96, Proposition 4.1] one can prove that conformal blocks divisors vanish above the theta level:*

**Proposition 4.4.3.** *Suppose that $\ell > \theta(\mathfrak{g}, \vec{\lambda})$, then $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)) = 0$.*

### 4.4.3 Applications

**Proposition 4.4.4.** *[BGM15b] Let $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{g})^n$, and suppose that $N_1, N_2, N_3, N_4$ is a partitionpartion of $[n] = \{1, \ldots, n\}$ into four nonempty subsets ordered so that if $\lambda(N_i) = \sum_{j \in N_i} |\lambda_j|$, then $\lambda(N_1) \leqslant \cdots \leqslant \lambda(N_4)$. If $\sum_{j \in \{1,2,3\}} \lambda(N_j) \leqslant \ell + 1$, then*

$$c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)) \cdot F_{N_1,N_2,N_3} = 0,$$

*and in particular, $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ is extremal in the nef cone.*

*The proof of Proposition 4.4.4 is analogous to that of Proposition 4.3.1.*

**Theorem 4.4.5.** *[BGM15b] Let $\mathbb{D} = \mathbb{D}_{\mathfrak{g},\vec{\lambda},\ell}$ be such that:*

1. *$0 < |\lambda_i| \leqslant \ell + 1$ for all $i \in \{1, \dots, n\}$;*
2. *$\sum_{i=1}^{n} |\lambda_i| > 2(\ell + 1)$.*

*Then the morphism $\phi_{\mathbb{D}}$ factors through $\rho_{\mathcal{A}} : \overline{\mathrm{M}}_{0,n} \longrightarrow \overline{\mathrm{M}}_{0,\mathcal{A}}$, where $\mathcal{A} = \{a_1, \dots, a_n\}$, $a_i = \frac{|\lambda_i|}{\ell+1}$.*

*The proof is analogous to the proof of Theorem 4.3.4.*

### 4.4.4 Examples comparing the theta and critical levels

**Example 4.4.6.** *The bundle $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_3, \omega_1^6, 1)$ on $\overline{\mathrm{M}}_{0,6}$ is at the critical level and so by Theorem 4.2.2, we have*

1. *$c_1(\mathbb{V}(\mathfrak{sl}_3, \omega_1^6, 1)) = c_1(\mathbb{V}(\mathfrak{sl}_2, \omega_1^6, 2))$; and*
2. *$c_1(\mathbb{V}(\mathfrak{sl}_3, \omega_1^6, 1 + c)) = c_1(\mathbb{V}(\mathfrak{sl}_2, \omega_1^6, 2 + c)) = 0$ for all $c \geqslant 1$.*

*The critical and theta levels for $\mathfrak{sl}_2$ bundles are equal, but the theta level for the bundle $\mathbb{V}$ is $2$, and so in this case, CL-vanishing is stronger than $\theta$-level vanishing.*

**Example 4.4.7.** *The bundle $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_3, \{2\omega_1 + \omega_2, \omega_2, 2\omega_1, 2\omega_2, 3\omega_2\}, 5)$ on $\overline{\mathrm{M}}_{0,5}$ is at the critical level and so by Theorem 4.2.2, we have*

1. *$c_1(\mathbb{V}) = c_1(\mathbb{V}(\mathfrak{sl}_6, \{\omega_1 + \omega_3, 2\omega_1, \omega_2, 2\omega_2, 2\omega_3\}, 2))$; and*
2. *$c_1(\mathbb{V}(\mathfrak{sl}_3, \{2\omega_1+\omega_2, \omega_2, 2\omega_1, 2\omega_2, 3\omega_2\}, 5+c)) = c_1(\mathbb{V}(\mathfrak{sl}_6, \{\omega_1+\omega_3, 2\omega_1, \omega_2, 2\omega_2, 2\omega_3\}, 2+c)) = 0$ for all $c \geqslant 1$.*

*In this case, for the $\mathfrak{sl}_3$ bundle $\mathbb{V}$, the theta level is $4.5 < 5$, and so in fact, by Theorem 4.4.3, $c_1(\mathbb{V}) = c_1(\mathbb{V}(\mathfrak{sl}_6, \{\omega_1 + \omega_3, 2\omega_1, \omega_2, 2\omega_2, 2\omega_3\}, 2)) = 0$, and in this case, $\theta$-level vanishing is stronger than CL vanishing.*

*The upshot is that except for $\mathfrak{sl}_2$, when Theta-level and Critical-level are the same, the vanishing results give different information about the bundles.*

## 4.5 The problem of nonvanishing

*Recall that we know at least three circumstances during which a conformal blocks divisor $c_1(\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell))$ will be zero:*

1. *If* $R = \mathrm{Rk}(\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)) = \dim(\mathrm{A}(\mathfrak{g},\vec{\lambda})) = A$*, then the contraction morphism given by* $c_1(\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell))$ *has image equal to a point:*
$$\overline{\mathrm{M}}_{0,n} \to \mathrm{Grass}^{\mathrm{quo}}(\mathrm{A}(\mathfrak{g},\vec{\lambda}),R) \overset{Plücker}{\hookrightarrow} \mathbb{P}^{\binom{A}{R}-1} = pt.$$
$$(C,\vec{p}) \mapsto [\mathrm{A}(\mathfrak{g},\vec{\lambda}) \twoheadrightarrow \mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(C,\vec{p})}] \mapsto [\Lambda^A\, \mathrm{A}(\mathfrak{g},\vec{\lambda}) \twoheadrightarrow \Lambda^R\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(C,\vec{p})}].$$
2. *If* $\ell > \theta(\mathfrak{g},\vec{\lambda}) = -1 + \frac{\sum_{i=1}\ell(\lambda_i)}{2}$*; or*
3. *In case* $\mathfrak{g} = \mathfrak{sl}_{r+1}$*, if If* $\ell > c(\mathfrak{sl}_{r+1},\vec{\lambda}) = -1 + \frac{\sum_{i=1}|\lambda_i|}{r+1}$.

*As it turns out,* $c_1(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)) \neq 0$ *as long as*
$$1 \leqslant \ell \leqslant cl(\mathfrak{sl}_2,\vec{\lambda}) = \theta(\mathfrak{sl}_2,\vec{\lambda}), \;\; and \;\; \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)) > 0.$$

*As we shall see today, a similar result holds when the critical and theta levels coincide. This is not the case generally, as we see in many examples even for* $\mathfrak{sl}_4$*. For example, the bundle* $\mathbb{V}(\mathfrak{sl}_4,\{\omega_2+\omega_3,\omega_1,\omega_1+2\omega_2,2\omega_1+\omega_3\},3)$ *is at the critical level, and it is below the theta level (which is 3.5). The rank of* $\mathbb{V}_{\mathfrak{sl}_4,\{\omega_1,(2\omega_1+\omega_3)^3\},3}$ *on* $\overline{\mathrm{M}}_{0,4}$ *is one, while the dimension of the vector space of coinvariants* $\mathbb{A}_{\mathfrak{sl}_4,\{\omega_1,(2\omega_1+\omega_3)^3\}}$ *is 2. A calculation shows that* $\mathbb{D}_{\mathfrak{sl}_4,\{\omega_1,(2\omega_1+\omega_3)^3\},3} = 0$.

*Examples like this have led us in [BGM15a] to ask when divisors are nonzero.*

**Question 4.5.1.** *What are necessary and sufficient conditions for a triple* $(\mathfrak{g},\vec{\lambda},\ell)$ *that guarantee that the associated conformal blocks divisor* $\mathbb{D}_{\mathfrak{g},\vec{\lambda},\ell}$ *is nonzero?*

*One approach, is to decompose a vector bundle into simpler bundles, whose vanishing may be understood more readily, and I will present today an additive identity (Proposition 4.6.1) dependent on ranks. For instance, one can decompose the divisor above as the following sum*
$$\mathbb{D}_{\mathfrak{sl}_4,\{\omega_1,2\omega_1+\omega_3,2\omega_1+\omega_3,2\omega_1+\omega_3\},3} = \mathbb{D}_{\mathfrak{sl}_4,\{\omega_1,\ldots,\omega_1\},1} + \mathbb{D}_{\mathfrak{sl}_4,\{0,\omega_1+\omega_3,\omega_1+\omega_3,\omega_1+\omega_3\},2}.$$

*Both of the divisors on the right hand side turn out to be trivial: the first since it is above the critical level, and the second, because it is pulled back from* $\overline{\mathrm{M}}_{0,3}$.

*We'll also see another type of identity in type A, where we decompose the Lie algebra and the weights. This gives a non-vanishing result in case the critical and theta levels coincide, such as the* $\mathfrak{sl}_2$ *result mentioned earlier. To prove the second identity one uses an interpretation of conformal blocks in terms of generalized theta functions.*

## 4.6 Additive identities dependent on ranks

*I will explain the following criteria, given in [BGM15a] for decomposing a divisor as an effective sum of simpler conformal blocks divisors.*

**Proposition 4.6.1.** *Let $\vec{\mu} \in P_{\ell}(\mathfrak{g})^n$, and $\vec{\nu} \in P_m(\mathfrak{g})^n$ be two n-tuple of dominant weights such that* $\operatorname{rk} \mathbb{V}_{\mathfrak{g},\vec{\mu},\ell} = 1$, *and* $\operatorname{rk} \mathbb{V}_{\mathfrak{g},\vec{\mu}+\vec{\nu},\ell+m} = \operatorname{rk} \mathbb{V}_{\mathfrak{g},\vec{\nu},m} = \delta$. *Then*

$$\mathbb{D}_{\mathfrak{g},\vec{\mu}+\vec{\nu},\ell+m} = \delta \cdot \mathbb{D}_{\mathfrak{g},\vec{\mu},\ell} + \mathbb{D}_{\mathfrak{g},\vec{\nu},m}.$$

*Before giving an outline of the proof of Proposition 4.6.1, I'll give some of the applications we showed in [BGM15a].*

*Using Proposition 4.6.1 in conjunction with the quantum generalization of a conjecture of Fulton in invariant theory [Bel07] and [BK16, Remark 8.5], we show in Corollary 4.6.2 that if* $\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell) = 1$, *then*

$$\mathbb{D}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell) = N \cdot \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell), \ \ \forall N \in \mathbb{N}.$$

*As an application of this, one can identify images of the maps $\phi_{\mathbb{D}}$ for $\mathbb{D} = \mathbb{D}(\mathfrak{sl}_{r+1}, \ell\vec{\lambda}, \ell) = \ell\, \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, 1)$, as the generalized Veronese quotients of [Gia13, GJM13].*

*Proposition 4.6.1 can be used to show that a divisor is nontrivial, by writing it as an effective sum of simpler divisors, and then showing one of the summands is nontrivial.*

*In [BGM15a], we use Proposition 4.6.1 to give non-trivial conformal blocks divisors, with non-zero weights, that do not give birational morphisms. Such examples were not known before. One may also approach questions of mysterious vanishing in this way, seeing for example a divisor as a sum of divisors whose vanishing can be explained by other means.*

*This result enables one to simplify questions of vanishing of a particular divisor into problems about its simpler constituent parts. But there have been more applications as well. For example, in ([Kaz16], Theorem 1.1) this result was used to prove that any $S_n$-invariant divisor for $\mathfrak{sl}_n$ on $\overline{M}_{0,n}$ coming from a bundle of rank one was in fact a sum of level one divisors in type A. In particular, the cone generated by infinitely many such divisors is finitely generated.*

*Fulton conjectured that if* $\operatorname{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda})) = 1$ *then* $\operatorname{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, N\vec{\lambda})) = 1$ $\forall\ N \in \mathbb{Z}_{>0}$. *This was proved by Knutson, Tao and Woodward [KTW04]. The quantum generalization of Fulton's conjecture [Bel07, BK16] is the following: Suppose* $\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell) = 1$ *($\ell$ is not necessarily the critical level) then* $\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell)) = 1$ *for all positive integers $N$. Using this generalization and Proposition 4.6.1, we obtain (by induction):*

**Corollary 4.6.2.** *If* $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = 1$, *then* $\mathbb{V}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell)) = N\ \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell))$, $\forall$ $N \in \mathbb{Z}_{>0}$.

**Remark 4.6.3.** *Corollary 4.6.2 appears in case* $r = 1$ *and* $\vec{\lambda} = (\omega_1, \ldots, \omega_1)$ *in ([GJMS13], Proposition 5.2). An analogous result for* $\mathfrak{g} = \mathfrak{so}_{2r+1}$ *appears in the work of Mukhopadhyay.*

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