Combinatorial aspects of cones of positive cycles on the moduli space of curves

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Abstract

These are notes from two lectures I gave at the Fields Institute during the major thematic program on Combinatorial Algebraic Geometry, for which I was asked to be prepared to talk on combinatorial aspects of $\overline{\mathrm{M}}_g$.

The scientific aims of the overall program were to: (1) introduce the study of ”combinatorial varieties” to the mathematical community as a thematically unified whole, (2) refine the techniques used within algebraic geometry to study combinatorial varieties, and (3) enlarge the class of algebraic spaces which have a recognized combinatorial structure. The organizers told us that they hoped that the Introductory Workshop would help to define Combinatorial Algebraic Geometry as a coherent subfield.

I must say that I found the task of writing two talks on combinatorial aspects of the moduli space of curves to be paralyzing, mainly due to the incredible amount of work on $\overline{\mathrm{M}}_g$ of this flavor. There are a number of groups of researchers whose work on the moduli of curves or related spaces I would say could be classified as combinatorial.

In the end I decided to frame my discussion around two problems which came from a comparison of $\overline{\mathrm{M}}_g$ with toric varieties, one of which has recently been solved (nearly at least), another which remains stubbornly open, and questions that have emerged from these. This gave me the opportunity to introduce vector bundles of conformal blocks, which are defined on the stacks $\overline{\mathcal{M}}_{g,n}$. For $g = 0$, the bundles are globally generated, and therefore their Chern classes have positivity properties: first Chern classes are base point free, and higher Chern classes give elements of Fulger and Lehman's Pliant cones. There are combinatorial aspects of vector bundles of conformal blocks, and many open problems about them, some of which I was able to discuss briefly at Fields.

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Lecture 1

Introduction to combinatorial $\overline{\mathrm{M}}_g$

Moduli spaces of curves occupy a distinguished position in algebraic geometry. As moduli spaces, they give insight into the study of smooth curves and their degenerations, they have played a principal role as a prototype for moduli of higher dimensional varieties [KSB88, Ale02, HM06, HKT06, HKT09, CGK09]. As special varieties, they have been one of the chief concrete, nontrivial settings where the nuanced theory of the minimal model program has been exhibited and explored [HH09, HH13, AFSvdW16, AFS16a, AFS16b].

It is not uncommon to refer to certain varieties as combinatorial: these include (but of course are not limited to) toric varieties: like projective space, weighted projective spaces, and certain blowups of those, Grassmannian varieties, or even more generally homogeneous varieties. These all come with group actions, and combinatorial data encoded in convex bodies keeps track of their important geometric features. Certain varieties like the moduli space of curves, have combinatorial structures reminiscent of varieties that are more traditionally considered to be combinatorial. As a result, various analogies have been made between them and the moduli of curves. Such comparisons have led to questions and conjectures, surprising formulas, and even arguments that have been used to detect and to prove some of the most important and often subtle geometric properties of the moduli space of curves.

I will try to illustrate the combinatorial nature of the moduli space of curves with examples close to my own experience. As you can see from other more complete surveys [Har84, Far09, Abr13, Cos10], this is a long studied subject with many points of focus!

1.1 What is M_g and why do we we want to compactify it?

Points in M_g correspond to isomorphism classes of smooth curves of genus g.

If C is a smooth curve (a 1-dimensional scheme over an algebraically closed field k), then its genus is

$$g = \dim \mathrm{H}^0(C, \omega_C) = \dim \mathrm{H}^1(C, \mathcal{O}_C),$$

where ω_C is the sheaf of regular 1-forms on C. If $k = \mathbb{C}$, then C is a smooth compact Riemann surface, and the algebraic definition of genus is the same as the topological definition.

Given any flat family $\mathcal{F} \to B$ of curves of genus g, there is a morphism $B \to \mathrm{M}_g$, that takes a point in the base b to the isomorphism class of the fiber. In fact M_g is something called a coarse moduli space. Because every curve with automorphisms can be used to construct a nontrivial family whose fibers are all isomorphic, one can show that M_g is not a fine moduli space. For these definitions, and some examples, see the Chapter 3.

Intuitively, smooth curves degenerate to singular ones. For example, we can write down the "general curve of genus 2" using the equation:

$$y^2 = x^6 + a_5x^5 + a_4x^4 + \cdots + a_1x + a_0.$$

A general point $(a_0, \ldots, a_5) \in \mathbb{A}^6$ determines a smooth curve. In other words, there is a family of curves parametrized by an open subset of $\mathbb{A}^6$, that includes the general smooth curve of genus 2. Certainly you can see that as the coefficients change the curves will sometimes have singularities.

To usefully parametrize families of curves like this one, it really pays to work with a proper space that parametrizes curves that have singularities. The space we will talk about today is denoted $\overline{\mathrm{M}}_g$, and it parametrizes stable curves of genus g.

Definition 1.1.1. *A* ***stable curve*** *C of (arithmetic) genus g is a reduced, connected, one dimensional scheme such that*

1. *C has only ordinary double points as singularities.*

2. *C has only a finite number of automorphisms.*

Remark 1.1.2. *To say that C has only a finite number of automorphisms, comes down to requiring that if C_i is a nonsingular rational component, C_i meets the rest of the curve in at least three points, and if C_i is a component of genus one, then it meets the rest of the curve in at least one point.*

Definition 1.1.3. *Let $\overline{\mathrm{M}}_g$, the moduli space of stable curves of genus g be the variety whose points are in one-to-one correspondence with isomorphism classes of stable curves of genus g.*

That such a variety $\overline{\mathrm{M}}_g$ exists is nontrivial. This was proved by Deligne and Mumford who constructed $\overline{\mathrm{M}}_g$ using Geometric Invariant Theory [DM69]. There are other choices of compactifications of M_g, and some of these compactifications receive birational morphisms from $\overline{\mathrm{M}}_g$; other compactifications just receive rational maps from $\overline{\mathrm{M}}_g$.

One could for example, consider the compactification of M_g by the moduli space $\overline{\mathrm{M}}_g^{ps}$ of pseudo-stable curves.

Definition 1.1.4. *A complete connected curve is pseudo stable if*

1. *it is reduced, and has at worst simple nodes and cuspidal singularities;*
2. *Every subcurve of genus 1 meets the rest of the curve in at least two points;*
3. *Every subcurve of genus zero meets the rest of the curve in at least three points.*

Replacing elliptic tails with cusps gives the morphism

$$T : \overline{\mathrm{M}}_g \longrightarrow \overline{\mathrm{M}}_g^{ps}.$$

The morphism T is an isomorphism outside of a codimension one locus Δ_1, whose generic point $[C] \in \overline{\mathrm{M}}_g$ is a curve C with a single separating node, whose normalization is a curve of genus one and a curve of genus $g-1$.

We can keep track of this information using cones of divisors and curves in the Neron Severi space, which I will take a moment to define.

1.2 Cones of divisors

Let X be a projective, not necessarily smooth variety defined over an algebraically closed field. Good references for the concepts below are [Laz04a, Laz04b].

Definition 1.2.1. *A variety X is called $\mathbb{Q}$-factorial if every Weil divisor on X is $\mathbb{Q}$-Cartier. We assume today that X is a $\mathbb{Q}$-factorial normal, projective variety over the complex numbers. The moduli spaces $\overline{\mathrm{M}}_{g,n}$ have these properties.*

Definition 1.2.2. *Two divisors D_1 and D_2 are numerically equivalent, written $D_1 \equiv D_2$, if they intersect all irreducible curves in the same degree. We say two curves C_1 and C_2 are numerically equivalent, written $C_1 \equiv C_2$ if $C_1 \cdot D = C_2 \cdot D$ for every irreducible subvariety D of codimension one in X.*

Definition 1.2.3. *We set $N_1(X)_{\mathbb{Z}}$ equal to the vector space of curves up to numerical equivalence, and $N^1(X)_{\mathbb{Z}}$ equal to the vector space of divisors up to numerical equivalence, and set*

$$\mathrm{N}^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \;\; \mathrm{N}^1(X) = \mathrm{N}^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R},$$

and

$$\mathrm{N}_1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \;\; \mathrm{N}_1(X) = \mathrm{N}_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The nef and pseudo-effective cones on X are subcones of vector spaces $\mathrm{N}^k(X)$, and $\mathrm{N}_k(X)$, which can be define analogously, and which I define for arbitrary proper varieties in Section 3.3.1. This perspective involves thinking about cycles as being naturally dual to Chern classes of vector bundles.

Definition 1.2.4. *The pseudo effective cone $\overline{\mathrm{Eff}}_k(X) \subset \mathrm{N}_k(\overline{\mathrm{M}}_{g,n})$ is defined to be the closure of the cone generated by k-cycles with nonnegative coefficients. Similarly $\overline{\mathrm{Eff}}^k(X) \subset \mathrm{N}^k(X)$ is defined to be the closure of the cone generated by cycles of codimension k with nonnegative coefficients.*

The cones $\overline{\mathrm{Eff}}_k(X)$, and $\overline{\mathrm{Eff}}^k(X)$ are full dimensional, spanning the vector spaces $\mathrm{N}_k(X)$, and $\mathrm{N}^k(X)$. They are pointed (containing no lines), closed, and convex.

Definition 1.2.5. *The Nef Cone $\mathrm{Nef}^k(X) \subset \mathrm{N}^k(X)$ is the cone dual to $\overline{\mathrm{Eff}}_k(X)$.*

As the dual of $\overline{\mathrm{Eff}}_k(X)$, the nef cone has all of the nice properties that $\overline{\mathrm{Eff}}_k(X)$ does.

The nef cone can also be defined as the closure of the cone generated by semi-ample divisors – divisors that correspond to morphisms, and

$$f : X \to Y \;\text{ is a regular map, then }\; f^*(\mathrm{Nef}(Y)) \subset \mathrm{Nef}^1(X).$$

Given a projective variety Y, and a morphism $f : X \longrightarrow Y \hookrightarrow \mathbb{P}^N$, then for any ample divisor $A = \mathcal{O}(1)|_Y$ on Y, one has the pullback divisor $D = f^*A$ on X is base point free. In fact, this divisor D is not only base point free, it has the much weaker property that it is nef. For if C is a curve on our projective variety X, then by the projection formula

$$D \cdot C = f_*(D \cdot C) = A \cdot f_*C,$$

which is zero if the map f *contracts* C, and otherwise, as A is ample, it is positive.

It is not true that every nef divisor on an arbitrary proper variety X has an associated morphism; To have such a property would be very special (a dream situation). But as we saw above, the divisors that give rise to maps do live in the nef cone, and for that reason the nef cone can be used a tool to understand the birational geometry of the space.

Definition 1.2.6. *For a $\mathbb{Q}$-Cartier divisor D on a proper variety X, we define:*

- *the stable base locus of D to be the union (with reduced structure) of all points in X which are in the base locus of the linear series $|nmD|$, for all n, where m is the smallest integer $\geqslant 1$ such that mD is Cartier;*
- *A moving $\mathbb{Q}$-Cartier divisor to be a divisor whose stable base locus has codimension 2 or more; and*
- *the moving cone* $\mathrm{Mov}(X)$ *of X, is the closure of the cone of moving divisors.*

Sufficiently high and divisible multiples of any effective divisor D on X will define a rational map (although not necessarily a morphism) from X to a projective variety Y. The stable base locus of D is the locus where the associated rational map will not be defined. The pseudo-effective cone may be divided into chambers having to do with the stable base loci [ELM+06, ELM+09]. Moreover, if

$$f : X \dashrightarrow Y \text{ is a rational map, then } f^*(\mathrm{Nef}(Y)) \subset \mathrm{Mov}(X),$$

and we have

$$\mathrm{Nef}^1(X) \subseteq \mathrm{Mov}(X) \subseteq \overline{\mathrm{Eff}}^1(X).$$

1.3 Examples

I will start with a simple example to illustrate how even very crude information about the location of the cone of nef divisors with respect to the effective cone tells us valuable information about the geometry of the variety X, as we see for $\overline{\mathrm{M}}_g$. Then we will look at a chamber decomposition of the nef cone of $\overline{\mathrm{M}}_3$, as pictured on the poster for the semester program.

1.3.1 $\mathrm{Nef}^1(X) \subseteq \mathrm{Mov}(X) \subseteq \overline{\mathrm{Eff}}^1(X)$

Theorem 1.3.1. *Every nef divisor on $\overline{\mathrm{M}}_g$ is big. In particular, there are no morphisms, with connected fibers from $\overline{\mathrm{M}}_g$ to any lower dimensional projective varieties other than a point.*

Theorem 1.3.1 says that the nef cone of $\overline{\mathrm{M}}_g$ sits properly inside of the cone of effective divisors– and their extremal faces only touch at the origin of the Nerón Severi space.

The statement for pointed curves is a little bit more complicated, but still very simple in the grand scheme of things:

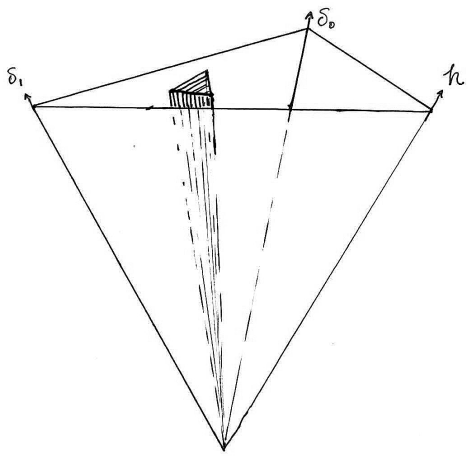


Figure 1.1: $\mathrm{Nef}^1(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$ with generators λ, $12\lambda - \delta_0$, and $10\lambda - \delta_0 - 2\delta_1$.

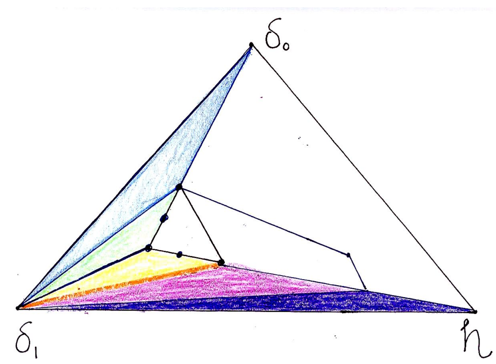


Figure 1.2: A partial chamber decomposition of

$$\mathrm{Nef}^1(\overline{\mathrm{M}}_3) \subset \mathrm{Mov}(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$$

seen in a cross section.

Theorem 1.3.2. *For $g \geqslant 2$, any nef divisor is either big or is numerically equivalent to the pullback of a big divisor by composition of projection morphisms. In particular, for $g \geqslant 2$, the only morphisms with connected fibers from $\overline{\mathrm{M}}_{g,n}$ to lower dimensional projective varieties are compositions of projections given by dropping points, followed by birational maps.*

1.3.2 A chamber decomposition for $\mathrm{Nef}(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$

The first work on these cones was done by Mumford in [Mum83], where everything was worked out for $\overline{\mathrm{M}}_2$, and where it was checked that the intersection theory could be done on $\overline{\mathrm{M}}_g$ in general. By [Fab90a], we know that $\overline{\mathrm{NE}}^1(\overline{\mathrm{M}}_3)$ is spanned by the classes $\delta_0 = [\Delta_0]$, $\delta_1 = [\Delta_1]$ and the class h of the hyperelliptic locus $\mathcal{H}_3$. The hyperelliptic locus $\mathcal{H}_g$ on $\overline{\mathrm{M}}_g$ is isomorphic to $\widetilde{\mathcal{M}}_{0,2g+2}$ under the map

$$h : \widetilde{\mathcal{M}}_{0,2g+2} \xrightarrow{\cong} \mathcal{H}_g \subseteq \overline{\mathrm{M}}_g,$$

given by taking a double cover branched at the marked points. For $g = 2$, the map is an isomorphism, for $g = 3$ the image has codimension one, and for $g \geqslant 4$ the image has higher codimension and isn't a divisor.

There is a partial chamber decomposition of $\mathrm{Nef}(\overline{\mathrm{M}}_3) \subset \mathrm{Mov}(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{NE}}^1(\overline{\mathrm{M}}_3)$. Two chambers have to do with different compactifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties: The classical Torelli map

$$\mathcal{M}_g \xrightarrow{t} \mathcal{A}_g,$$

which takes a smooth curve X of genus g to its Jacobian, doesn't extend to a morphism on $\overline{\mathrm{M}}_g$. But there are extensions to various compactifications of $\mathcal{A}_g$.

The Satake Chamber

Let $\overline{\mathcal{A}}_g^{Sat}$ be the Satake compactification of the moduli space $\mathcal{A}_g$. The classical Torelli map extends to a regular map

$$t^{Sat} : \overline{\mathrm{M}}_g \longrightarrow \overline{\mathcal{A}}_g^{Sat}.$$

This morphism is given by the divisor λ. In other words, $\lambda = (t^{Sat})^*(A)$, where A is an ample divisor $\overline{\mathcal{A}}_g^{Sat}$.

1.3.3 The 2nd Voronoi Chamber

We let $\overline{\mathcal{A}}_g^{Vor}$: be the toroidal compactification of $\mathcal{A}_g$ for the $2nd$ Voronoi fan. The Torelli map is known to extend to the regular map

$$\bar{t}_g : \overline{\mathrm{M}}_g \xrightarrow{t^{Sat}} \mathcal{A}_g^{Vor(2)}.$$

This morphism is given by a divisor which lies on the (interior of the) face of the nef cone spanned by λ and $12\lambda - \delta_0$.

The Shepherd-Barron Unknown (SBU) Chamber

There is a morphism

$$f : \overline{\mathrm{M}}_g \longrightarrow X,$$

given by the base point free extremal nef divisor $12\lambda - \delta_0$. As far as I know, there isn't a modular interpretation for X.

1.3.4 The Pseudo-Stable Chamber

Let $\overline{\mathrm{M}}_g^{ps}$ be the moduli stack of pseudo stable curves. Replacing elliptic tails with cusps gives the divisorial contraction

$$T : \overline{\mathrm{M}}_g \longrightarrow \overline{\mathrm{M}}_g^{ps}.$$

T is given by a divisor that lies on the face of the nef cone spanned by $12\lambda - \delta_0$ and $10\lambda - \delta_0 - 2\delta_1$.

The C-Stable Chamber

Let $\overline{\mathrm{M}}_g^{cs}$ be the moduli space of c-stable curves. Contracting elliptic bridges to tacnodes defines the small modification $\psi : \overline{\mathrm{M}}_g^{ps} \longrightarrow \overline{\mathrm{M}}_g^{cs}$, and composing with T defines a regular map

$$\overline{\mathrm{M}}_g \xrightarrow{T} \overline{\mathrm{M}}_g^{ps} \xrightarrow{\psi} \overline{\mathrm{M}}_g^{cs},$$

given by the extremal divisor $10\lambda - \delta_0 - 2\delta_1$.

1.3.5 The First Flip: H-Semistable Curves in the Moving Cone

We can also see the first flip: Let $\overline{\mathrm{M}}_g^{hs}$ be the moduli space of h-semistable curves. There is a morphism $\psi^+ : \overline{\mathrm{M}}_g^{hs} \longrightarrow \overline{\mathrm{M}}_g^{cs}$ which is a flip of ψ:

$$\begin{array}{ccccc} & & \overline{\mathrm{M}}_g & & \\ & \swarrow^{T} & \downarrow & & \\ \overline{\mathrm{M}}_g^{ps} & & & & (\overline{\mathrm{M}}_g^{ps})^+ = \overline{\mathrm{M}}_g^{hs} \\ & \searrow_{\psi} & \downarrow & \swarrow_{\psi^+} & \\ & & \overline{\mathrm{M}}_g^{cs}. & & \end{array}$$

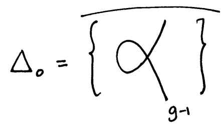
We can see the chamber of the effective cone of $\overline{\mathrm{M}}_3$ corresponding to $\overline{\mathrm{M}}_g^{hs}$. It doesn't touch the Nef cone of $\overline{\mathrm{M}}_3$ because there isn't a morphism from $\overline{\mathrm{M}}_3$ to $\overline{\mathrm{M}}_g^{hs}$. Instead, there is a rational map, which for $g = 3$ is given by the moving divisors pictured.

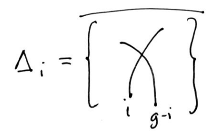
There is another chamber of the moving cone, as we can see in the picture. This corresponds to the pullback of the nef cone of the second flip.

1.4 The boundary $\overline{\mathrm{M}}_g \backslash \mathrm{M}_g$

1.4.1 The boundary described as a sublocus

The boundary of M_g consists of a union of components having at least one node. Components of the boundary come in two types, which which we may describe in a number of ways.





1. The components Δ_i can be described as having generic point with a separating node; the closure of the set of curves whose normalization consists of a pointed curve of genus i and a pointed curve of genus $g-i$; the image of the attaching map

$$\overline{\mathrm{M}}_{i,1} \times \overline{\mathrm{M}}_{g-i,1} \twoheadrightarrow \Delta_i,$$

given by attaching curves by gluing their marked points together.

2. The component $\Delta_0 = \Delta_{irr}$ can be described as having generic point with a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g-1$ with two marked points; and the image of the clutching map

$$\overline{\mathrm{M}}_{g-1,2} \twoheadrightarrow \Delta_0 = \Delta_{irr},$$

given by taking a curve of genus $g-1$ with two marked points to a nodal curve of genus g by gluing the two marked points together.

As one can see in the descriptions of the boundary components, moduli of pointed curves come up naturally even if one is only interested in studying $\overline{\mathrm{M}}_g$.

Definition 1.4.1. *A* ***stable n-pointed curve*** *is a complete connected curve* C *that has only nodes as singularities, together with an ordered collection* $p_1, p_2, \ldots, p_n \in C$ *of distinct smooth points of* C*, such that the* $(n+1)$*-tuple* $(C; p_1, \ldots, p_n)$ *has only a finite number of automorphisms.*

Definition 1.4.2. *For* $g = 0$*, let* $n \geqslant 3$*, and for* $g = 1$*, let* $n \geqslant 1$*:*

$$\overline{\mathcal{M}}_{g,n} : (\mathcal{S}ch_{\mathrm{k}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \overline{\mathcal{M}}_{g,n}(\mathrm{T}),$$

where $\overline{\mathcal{M}}_{g,n}(\mathrm{T})$ *is the set of proper families* $(\pi : \mathrm{X} \to \mathrm{T}; \{\sigma_i : T \to \mathrm{X}\}_{i=1}^n)$ *such that the fiber* $(\mathrm{X}_t, \{\sigma_i(t)\}_{i=1}^n)$*, at every geometric point* $t \in \mathrm{T}$ *is a stable n-pointed curve of genus g modulo isomorphism over* T*.*

Theorem 1.4.3. [KM76, Knu83a, Knu83b] *There exists a coarse moduli space* $\overline{\mathrm{M}}_{g,n}$ *for the moduli functor* $\overline{\mathcal{M}}_{g,n}$*; it is a projective variety that contains* $\mathrm{M}_{g,n}$ *as a dense open subset. Moreover,* $\overline{\mathrm{M}}_{0,n}$ *is a smooth projective variety that is a fine moduli space for* $\overline{\mathcal{M}}_{0,n}$*.*

The components of the boundary of $\overline{\mathrm{M}}_g$ of codimension k have analagous descriptions:

Definition 1.4.4. $\delta^k(\overline{\mathrm{M}}_g) = \{[C] \in \overline{\mathrm{M}}_g | \ C$ *has at least* k *nodes* $\}$.

One can see that $\delta^k(\overline{\mathrm{M}}_g)$ is the union of irreducible components, each of which can be constructed as the image of attaching and clutching maps from products of moduli of marked point spaces.

This gives a stratification of the space determined by the topological type of the curves being parametrized. Not unlike the torus invariant fixed loci of a toric variety, this has led researchers to ask questions about the moduli space of curves to determine if has features in common with toric varieties. These analogies influence current guiding problems and some that have been recently solved.

1.4.2 The boundary described by weighted graphs

Given a curve C of genus g, its dual graph $\Gamma(C)$ has a vertex v_i corresponding to each irreducible component C_i. The graph is weighted: Each vertex v_i is assigned an integer weight $g(v_i)$ corresponding to the genus of C_i. Corresponding to each node, where a point of C_i meets a point of C_j, there corresponds an edge joining vertices v_i and v_j. The genus of $\Gamma(C)$ and of the curve C is given by the formula:

$$b_1(\Gamma(C), \mathbb{Z}) + \sum_{v \in V(\Gamma(C))} g(v).$$

The graph $\Gamma(C)$ and the curve C are stable if every vertex v of genus $g(v) = 0$ has valence at least 3, and every vertex v with $g(v) = 1$ has valence at least 1.

Each irreducible component of the boundary M_Γ corresponds to a type of weighted graph γ which itself is a moduli space of codimension

$$|E(\Gamma)| = \text{ the number of edges of } \Gamma.$$

and

$$\overline{\mathrm{M}}_g = \cup_{g(\Gamma)=g} \overline{\mathrm{M}}_\Gamma.$$

The closure $\overline{\mathrm{M}}_\Gamma \supset \mathrm{M}_{\Gamma'}$, if and only if $\Gamma' \to \Gamma$ is a contraction: these are maps, contracting an edge connecting vertices v_1 and v_2 with genera g_1 and g_2 giving a vertex with genus $g_1 + g_2$, or contracting a loop on a vertex with genus g resulting in a vertex with genus $g + 1$.

A number of generalizations of the moduli space of curves have arisen from this boundary stratification. For example, the moduli space of rooted trees of projective d-spaces [CGK09], described in Section **??**, and the moduli space of tropical curves, described in Section 3.2.1.

In my second lecture I will talk problems about $\overline{\mathrm{M}}_g$ itself that have come directly from combinatorial analogies inspired by the boundary stratification.

Lecture 2

Cones of positive cycles on $\overline{\mathrm{M}}_{g,n}$

Cones of positive cycles are combinatorial devices that encode geometric data about proper varieties. Such cones of divisors and curves are the customary, time-honored, long established, and even familiar tools of the minimal model program. As we're starting to learn, their higher codimension analogues can behave very differently.

For instance, for any proper variety X we know that $\mathrm{Nef}^1(X)$ is contained in the pseudoeffective cone of codimension one cycles $\overline{\mathrm{Eff}}^1(X)$. But, as was proved in [DELV10], if E is an elliptic curve with complex multiplication, then $\overline{\mathrm{Eff}}^k(E^r) \subsetneq \mathrm{Nef}^k(E^r)$ for $1 < k < r-1$. In [Ott15], an example was given of a variety X of lines on a very general cubic fourfold where $\overline{\mathrm{Eff}}^2(X) \subsetneq \overline{\mathrm{Nef}}^2(X)$. Nef cycles of higher codimension fail to satisfy other nice properties of nef divisors: For instance, the product of two nef cycles is not necessarily nef.

To more accurately capture the properties of cycles of higher codimension, Fulger and Lehmann have introduced three sub-cones: the Pliant cone, the base-point free cone, and the universally pseudoeffective cone. A lot of work, and many open problems are emerging [FL14, CC14, Ott15, CC15, LO16, CLO16]. For instance, there are explicit examples of full-dimensional subcones of the Pliant cone of $\overline{\mathrm{M}}_{0,n}$ in all codimension.

After a few basic definitions, I will discuss some of the questions that have come up about such cones of cycles on $\overline{\mathrm{M}}_{g,n}$.

2.1 The F-Conjecture

Recall from the first lecture that in $\overline{\mathrm{M}}_{g,n}$, the locus

$$\delta^k(\overline{\mathrm{M}}_{g,n}) = \{(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n} \ : \ C \ \text{has at least k nodes } \}$$

has codimension k. For each k, the set $\delta^k(\overline{\mathrm{M}}_{g,n})$ decomposes into irreducible component indexed by dual graphs Γ with k edges. Moreover, the closure of the component corresponding to Γ contains components consisting of curves whose corresponding dual graph Γ' contracts to Γ. The resulting stratification of the space is both reminiscent and analogous to the combinatorial structure determined by the torus invariant loci of a toric variety.

On a complete toric variety, every effective cycle of dimension k can be expressed as a linear combination of torus invariant cycles of dimension k. Fulton compared the action of the symmetric group S_n on $\overline{\mathrm{M}}_{0,n}$ with the action of an algebraic torus a toric variety. Following this analogy, he asked whether a variety of dimension k could be expressed as an effective combination of boundary cycles of that dimension. As $\overline{\mathrm{M}}_{0,n}$ is rational, of dimension $n-3$, this is true for points and cycles of codimension $n-3$. For the statement to be true for divisors, it would say that every effective divisor would be in the cone spanned by the boundary divisors. This was proved false by Keel [GKM02, page 4] and Vermeire [Ver02], who found effective divisors not in the convex hull of the boundary divisors. For the statement to be true for curves, it would say that the Mori cone of curves is spanned by irreducible components of $\delta^{n-4}(\overline{\mathrm{M}}_{0,n})$: whose dual graph is distinctive: the only vertex that isn't trivalent has valency four. In particular, these are all curves that can be described as images of attaching or clutching maps from $\overline{\mathrm{M}}_{0,4}$.

Of course this question could just as well be asked for higher genus, and Faber did this, proving the statement for $\overline{\mathrm{M}}_3$ and $\overline{\mathrm{M}}_4$ (see eg. [Fab90a, Intermezzo]).

In honor of Faber and Fulton, the numerical equivalence classes of the irreducible components of $\delta^{3g-4+n}(\overline{\mathrm{M}}_{g,n})$ are called F-Curves. One can ask the following question:

Question 2.1.1. *(The* F*-Conjecture* [GKM02]*) Is every effective curve numerically equivalent to an effective combination of* F*-Curves? Otherwise said, can one say that a divisor is nef, if and only if it nonnegatively intersects all the* F*-Curves?*

In [GKM02], we showed that in fact a positive solution to this question for S_g-invariant nef divisors on $\overline{\mathrm{M}}_{0,g+n}$ would give a positive answer for divisors on $\overline{\mathrm{M}}_{g,n}$. In particular, the birational geometry of $\overline{\mathrm{M}}_{0,g}$ controls aspects of the birational geometry of $\overline{\mathrm{M}}_g$. We know now that the answer to this question is true on $\overline{\mathrm{M}}_{0,n}$ for $n \leqslant 7$ [KM13], and on $\overline{\mathrm{M}}_g$ for $g \leqslant 24$ [Gib09].

2.2 The question of whether $\overline{\mathrm{M}}_{0,n}$ is a MDS

Another analogy between $\overline{\mathrm{M}}_{0,n}$ and toric varieties prompted Hu and Keel to ask whether $\overline{\mathrm{M}}_{0,n}$ is a so-called Mori Dream Space. We now know, due to the very recent work of Castravet and Tevelev, that this is not true in general. I'll define a Mori Dream Space and state the results of Castravet and Tevelev. To do so, we need first the definition of a so-called *small $\mathbb{Q}$-factorial modification* of X, defined as follows:

Definition 2.2.1. *Let X be a normal projective variety. A small $\mathbb{Q}$-factorial modification of X is a birational map*[1] $f : X \to Y$ *that is an isomorphism in codimension one (ie. is* small*) to a normal $\mathbb{Q}$-factorial projective variety Y. We refer to f as an SQM for short.*

Definition 2.2.2. *A normal projective variety X is called an MDS if:*

(a) *X is $\mathbb{Q}$-factorial and* $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$*;*

(b) $\mathrm{Nef}(X)$ *is generated by finitely many semi-ample line bundles;*

(c) *there is a finite collection of SQMs $f_i : X \to X_i$ such that each X_i satisfies* (1) *and* (2) *and* $\mathrm{Mov}(X)$ *is the union of $f_i^*(Nef(X_i))$.*

Extremely well behaved schemes, like toric and log Fano varieties, where the minimal model program can be carried out without issue, were deemed "Mori Dream Spaces" by Hu and Keel (MDS for short). The moduli space of stable n-pointed genus zero curves $\overline{M}_{0,n}$ is Fano for $n \leqslant 6$, and so is a MDS in that range. While not Fano for $n \geqslant 7$, a comparison between the stratification of $\overline{M}_{0,n}$, given by curves according to topological type, to the stratification of a toric variety given by its torus invariant sub-loci, prompted Hu and Keel to ask whether $\overline{M}_{0,n}$ is a MDS for all n. This question has resulted in a great deal of work in the literature both about $\overline{M}_{0,n}$ and related spaces. As Castravet and Tevelev point out in their paper, for about 15 years now, many researchers have tried to understand this particular problem. Other related questions go back to the work of Mumford.

[1] In particular, this map f need not be regular.

Castravet and Tevelev in [CT15], prove that $\overline{M}_{0,n}$ is not a MDS as long as n is at least 134. The authors assert that rather than compare $\overline{M}_{0,n}$ to a toric variety, one should rather think of it as the blow up of a toric variety – namely, the blow up of the Losev Manin space $\overline{LM}_n$ at *the identity of the torus*. Using their work, in [GK16], González and Karu showed $\overline{M}_{0,n}$ is not an MDS as long as n is at least 13. A very recent preprint of Hausen, Keicher, and Laface [HKL16] studies the blow-up of a weighted projective plane at a general point, giving criteria and algorithms for testing if the result is a Mori dream space. As an application, using the framework of Castravet and Tevelev, they show that $\overline{M}_{0,n}$ is not an MDS as long as $n \geqslant 10$. The three cases 7, 8, and 9 therefore seem to remain open, as far as I know.

2.2.1 What comes out of these questions?

In Castravet and Tevelev's proof that $\overline{M}_{0,n}$ is not a MDS, they ultimately show that the third criterion of the definition for a MDS (see Definition 2.2.2) fails. If the second condition in the definition for a MDS, the prediction is that the Nef cone of $\overline{M}_{0,n}$ should have a finite number of extremal rays, and that every nef divisor should be semi-ample. Moreover, if in the increasingly unlikely event that the F-Conjecture were to hold for $\overline{\mathrm{M}}_{0,n}$, then the Nef cone would have finitely many extremal rays. Therefore, it makes sense to ask:

Question 2.2.3. *(a) Is* $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$ *polyhedral?*

(b) Is every element of $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$ *semi-ample?*

It would be interesting to see that the answer to part (b) is yes, but that there are so many nef divisors that the answer to part (a) is no. This leads me to want to tell you about a class of very many globally generated vector bundles on $\overline{\mathrm{M}}_{0,n}$. In particular, their first Chern classes are all base point free elements of the nef cone, spanning a full dimensional subcone of semi-ample divisors in $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$.

2.3 Vector bundles of conformal blocks

The stack $\overline{\mathcal{M}}_{g,n}$, parametrizing flat families of stable n-pointed curves of genus g, carries vector bundles $\mathbb{V}$, constructed using representation theory [TUY89]. Over (closed) points $(C, \vec{p}) \in \overline{\mathcal{M}}_{g,n}$, fibers of $\mathbb{V}$ are dual to vector spaces of conformal blocks, basic objects in rational conformal field theory. When $g = 0$, the bundles are globally generated, and (products of) higher Chern classes of these bundles are elements in the pliant cone on the moduli *space* $\overline{\mathrm{M}}_{0,n}$.

These are combinatorial in many ways, for example, in type A their ranks can be computed using Schubert calculus. Namely, the cohomological version of Witten's Dictionary, stated below, which follows from [Wit95] and the twisting procedure of [Bel08, Eq (3.10)].

Theorem 2.3.1. *Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ be a vector bundle on $\overline{\mathcal{M}}_{g,n}$ such that $\sum_{i=1}^{n} |\lambda_i| = (r+1)(\ell+s)$ for some integer s.*

(a) *If $s > 0$, then let $\lambda = \ell\omega_1$. The rank of $\mathbb{V}$ is the coefficient of $q^s\sigma_{\ell\omega_{r+1}}$ in the quantum product*
$$\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \star \sigma_{\lambda}^{s} \in \mathrm{QH}^*(\mathrm{Gr}(r+1, r+1+\ell)).$$

(b) *If $s \leqslant 0$, then the rank of $\mathbb{V}$ is the multiplicity of the class of a point $\sigma_{k\omega_{r+1}}$ in the product*
$$\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \cdots \cdot \sigma_{\lambda_n} \in H^*(\mathrm{Gr}(r+1, r+1+k)),$$
where $k = \ell + s$.

Examples of rank computations using Theorem 2.3.1 can be found in [BGM15, BGM16, Kaz16, Hob16] and [BGK15].

2.3.1 Construction of fibers

I follow [TUY89, Uen08, Bea96] and [Fak12] pretty closely in my notation and definitions.

Given a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of elements $\lambda_i \in \mathcal{P}_\ell(\mathfrak{g})$, one can define a vector bundle of conformal blocks $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$.

Finite dimensional situation:

Recall that to λ_i there corresponds a unique finite dimensional $\mathfrak{g}$-module V_{λ_i}. Set $\mathrm{V}_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots V_{\lambda_n}$ and define an action
$$\mathfrak{g} \times \mathrm{V}_{\vec{\lambda}} \to \mathrm{V}_{\vec{\lambda}} \ \ (g, v_1 \otimes \cdots v_n) \mapsto \sum_{i=1}^{n} v_1 \otimes \cdots v_{i-1} \otimes (g \cdot v_i) \otimes v_{i+1} \otimes \cdots v_n.$$

We write $[V_{\vec{\lambda}}]_{\mathfrak{g}}$ for the **space of coinvariants of** $V_{\vec{\lambda}}$: The largest quotient of $V_{\vec{\lambda}}$ on which $\mathfrak{g}$ acts trivially. That is, the quotient of $V_{\vec{\lambda}}$ by the subspace spanned by the vectors $X \cdot v$ where $X \in \mathfrak{g}$ and $v \in V_{\vec{\lambda}}$.

Let V and W be two $\mathfrak{a}$-modules. The space of coinvariants $[V \otimes W]_{\mathfrak{g}}$ is equal to the quotient of $V \otimes W$ by the subspace spanned by the elements of the form

$$X v \otimes w + v \otimes X w,$$

where $X \in \mathfrak{g}$, $v \in V$, and $w \in W$.

Infinite dimensional analogues:

Given a stable n-pointed curve $(C, \vec{p})$, to construct the fiber $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p}}$ we will use two new Lie algebras:

First, for each $i \in \{1, \ldots\}$ we will use

$$\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C} \cdot c,$$

where by $\mathbb{C}((\xi_i))$, we mean the field of Laurant power series over $\mathbb{C}$ in the variable ξ_i, and c is in the center of $\hat{\mathfrak{g}}_i$. To define the bracket, we note that elements in $\hat{\mathfrak{g}}_i$ are tuples $(a_i, \alpha c)$, with $a_i = \sum_j X_{ij} \otimes f_{ij}$, with $f_{ij} \in \mathbb{C}((\xi_i))$. We define the bracket on simple tensors:

$$[(X \otimes f, \alpha c), (Y \otimes g, \beta c)] = ([X, Y] \otimes fg, c(X, Y) \cdot \mathrm{Res}_{\xi_i = 0}(g(\xi_i) df(\xi_i))).$$

Checking $\hat{\mathfrak{g}}_i$ is a Lie algebra done in Section 3.4, where we also outline the construction of the infinite dimensional analogue H_{λ_i} of V_{λ_i}: It turns out that H_{λ_i} is a unique $\hat{\mathfrak{g}}_i$-module, although infinite dimensional.

Now for the second Lie algebra:

Let $U = C \setminus \{p_1, \ldots, p_n\}$. Without loss of generality, we can assume that U is affine since we can always add marked points with trivial weights (this is nontrivial). By $\mathfrak{g}(U)$ we mean the Lie algebra $\mathfrak{g} \otimes \mathcal{O}_C(U)$.

Choose a local coordinate ξ_i at each point p_i, and denote by f_{p_i} the Laurant expansion of any element $f \in \mathcal{O}_C(U)$. Then for each i, we get a ring homomorphism

$$\mathcal{O}_C(U) \to \mathbb{C}((\xi_i)), \quad f \mapsto f_{p_i},$$

and hence for each i, we obtain a map (this is not a Lie algebra embedding)

$$\mathfrak{g}(U) \to \hat{\mathfrak{g}}_i \quad \mathrm{X} \otimes f \mapsto (\mathrm{X} \otimes f_{p_i}, 0).$$

Set $\mathrm{H}_{\vec{\lambda}} = H_{\lambda_1} \otimes \cdots H_{\lambda_n}$ and define the following, which we will show is an action:

$$\mathfrak{g}(U) \times \mathrm{H}_{\vec{\lambda}} \to \mathrm{H}_{\vec{\lambda}} \quad (g, w_1 \otimes \cdots w_n) \mapsto \sum_{i=1}^{n} w_1 \otimes \cdots w_{i-1} \otimes (g \cdot w_i) \otimes w_{i+1} \otimes \cdots w_n. \tag{2.1}$$

Claim 2.3.2. *Equation 2.1 defines an action of* $\mathfrak{g}(U)$ *on* $\mathrm{H}_{\vec{\lambda}}$.

Proof. Given $X \otimes f$, and $Y \otimes g \in \mathfrak{g}(U)$, and a simple tensor $v = v_1 \otimes \cdots \otimes v_n \in \mathrm{H}_{\vec{\lambda}}$, we want to check that

$$[X \otimes f, Y \otimes g] \cdot v = (X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big).$$

The right hand side simplifies as follows:

$$\begin{aligned}
&(X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big) \\
&\quad = (X \otimes f) \cdot \Big(\sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\Big) \\
&\quad - (Y \otimes g) \cdot \Big(\sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\Big) \\
&= \Big(\sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} v_1 \otimes \cdots v_{j-1} \otimes (X \otimes f_{p_j}) \cdot v_j \otimes v_{j+1} \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\Big) \\
&- \Big(\sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} v_1 \otimes \cdots v_{j-1} \otimes (Y \otimes g_{p_j}) \cdot v_j \otimes v_{j+1} \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\Big) \\
&\quad = \Big(\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot \big((Y \otimes g_{p_i}) \cdot v_i\big) \otimes v_{i+1} \otimes \cdots \otimes v_n\Big) \\
&\quad - \Big(\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot \big((X \otimes f_{p_i}) \cdot v_i\big) \otimes v_{i+1} \otimes \cdots \otimes v_n\Big) \\
&\qquad = \Big(\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes \big([X, Y] + (fg)_{p_i}\big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\Big)
\end{aligned} \tag{2.2}$$

The left hand side simplifies as follows:

(2.3)
$$\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes\Big([X,Y]\otimes f_{p_i}g_{p_i}+(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}c\Big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n$$
$$=\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes\Big([X,Y]\otimes f_{p_i}g_{p_i}\Big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n$$
$$+\sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes\Big((X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}c\Big)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n.$$

Now, by definition, $c\cdot v_i=\ell\cdot v_i$ for all i, and so we can rewrite the second summand as follows

$$(2.4)\quad \sum_{1\leqslant i\leqslant n} v_1\otimes\cdots\otimes v_{i-1}\otimes((X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}c)\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n$$
$$=\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}\Big(v_1\otimes\cdots\otimes v_{i-1}\otimes c\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)$$
$$=\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}\Big(v_1\otimes\cdots\otimes v_{i-1}\otimes \ell\cdot v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\Big)$$
$$=\big(\ell\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}\big)\Big(v_1\otimes\cdots\otimes v_n\Big).$$

Since $\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i}df_{p_i}=0$, this contribution is zero. Therefore the left and right hand sides of the expressions are the same, and we have checked that $\mathfrak{g}(U)$ acts on $\mathrm{H}_{\vec{\lambda}}$ as claimed. □

We now set
$$\mathbb{V}(\mathfrak{g},\vec{\lambda},\ell)|_{(C,\vec{p})}=[\mathrm{H}_{\vec{\lambda}}]_{\mathfrak{g}(U)}.$$

2.3.2 Vanishing, identities and the problem of nonvanishing

Critical level vanishing and identities

The critical level, first defined by Fakhrudin [?Fakh] for $\mathfrak{sl}_2$, is defined only for $\mathfrak{g}=\mathfrak{sl}_{r+1}$, while a similar concept called the theta level is defined for general Lie algebras $\mathfrak{g}$ [BGM15, BGM16]. As I will explain, the Chern classes of bundles are trivial if ℓ is above the critical level. In terms of first Chern classes, it seems that very many conformal blocks divisors are extremal in the nef cone, and the number of curves they contract increases as the level increases with respect to the pair $(\mathfrak{g}, \vec{\lambda})$. Moreover, sets of nontrivial classes where the Lie algebra and the weights are fixed but the level varies, have been shown to have interesting properties. For example on $\overline{\mathrm{M}}_{0,n}$, where $n = 2(g+1)$ is even $\{c_1(\mathbb{V}(\mathfrak{sl}_2, \omega_1^n, \ell)) : 1 \leqslant \ell \leqslant g = cl((\mathfrak{sl}_2, \omega_1^n)\}$, forms a basis of $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{S_n}$ [?ags].

Definition 2.3.3. *If $r+1$ divides $\sum_{i=1}^n |\lambda_i|$, we refer to*

$$cl(\mathfrak{sl}_{r+1}, \vec{\lambda}) = -1 + \frac{\sum_{i=1}^n |\lambda_i|}{r+1},$$

as the critical level for the pair $(\mathfrak{sl}_{r+1}, \vec{\lambda})$. If $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, and if $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})^n$, then $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ is called a critical level bundle, and $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ is called a critical level divisor.

Note that if $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, then $r = cl(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T)$, where $\vec{\lambda}^T = (\lambda_1^T, \ldots, \lambda_n^T)$. Here λ_i^T is the weight associated to the transpose of the Young diagram associated to the weight λ_i. In particular, $|\lambda_i| = |\lambda_i^T|$, and so

$$\sum_{i=1}^n |\lambda_i| = (r+1)(\ell+1) = (\ell+1)(r+1) = \sum_{i=1}^n |\lambda_i^T|.$$

In particular, critical level bundles come in pairs:

The following theorem was first proved by Fakhruddin for $\mathfrak{sl}_2$ in [?Fakh]:

Theorem 2.3.4. [?Fakh, BGM15] *If $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, then*

(a) $c_k(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell + c)) = 0$*, for $c \geqslant 1$; and*

(b) $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T, r))$.

Examples

(a) The bundle $\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)$ is at the critical level for $n = (r+1)(\ell+1)$. In [BGM15] we showed that the first Chern classes are all nonzero, and by Theorem 2.3.4, for $n = (r+1)(\ell+1)$,

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r)); \text{ and}$$

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell + c)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r + c)) = 0 \text{ for all } c \geqslant 1.$$

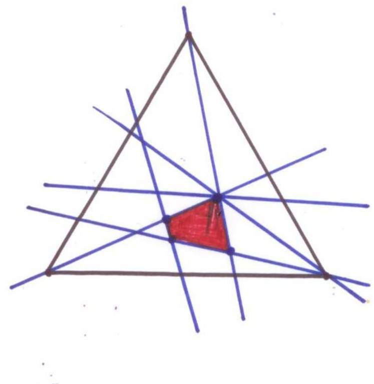


Figure 2.1:

$$\text{Nef}^1(\overline{\text{M}}_{0,8})^{S_8} \subset \overline{\text{Eff}}^1(\overline{\text{M}}_{0,8})^{S_8}$$

(b) The bundle $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, (2\omega_1 + \omega_3)^3\}, 3)$ is at the critical level, and its first Chern class is self dual.

Remark 2.3.5. *The main applications of vanishing above the critical level are extremality tests, and criteria for showing that maps given by conformal blocks divisors factor through contraction maps to Hassett spaces.*

The problem of nonvanishing

The bundle $\mathbb{V}(\mathfrak{sl}_4, \{\omega_2 + \omega_3, \omega_1, \omega_1 + 2\omega_2, 2\omega_1 + \omega_3\}, 3)$ is at the critical level, (and it is below the theta level (which is 3.5)). The rank of $\mathbb{V}_{\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}, 3}$ on $\overline{\text{M}}_{0,4}$ is one, while the dimension of the vector space of coinvariants $\mathbb{A}_{\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}}$ is 2. A calculation shows that $\mathbb{D}_{\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}, 3} = 0$.

Examples like this have led us in [BGM16] to ask when divisors are nonzero.

Question 2.3.6. *What are necessary and sufficient conditions for a triple $(\mathfrak{g}, \vec{\lambda}, \ell)$ that guarantee that the associated conformal blocks divisor $\mathbb{D}_{\mathfrak{g}, \vec{\lambda}, \ell}$ is nonzero?*

2.4 Cones of cycles of higher codimension

2.4.1 The Pliant cone of Lehmann and Fulger

As discussed earlier, the pseudo-effective cone $\overline{\mathrm{Eff}}_m(\mathrm{X})$ is the closure of the cone generated by classes of m-dimensional subvarieties on a projective variety X. If X is smooth, then one can define higher codimension analogues of cones of nef divisors by taking $\mathrm{Nef}^m(\mathrm{X})$ to be dual to $\overline{\mathrm{Eff}}_m(\mathrm{X})$. Many properties held by these cones when $m = 1$ fail more generally [Pet09, Voi10, DELV10, FL14]. To more accurately capture the properties of cones of nef divisors, Fulger and Lehmann have introduced three sub-cones: the Pliant cone, the base-point free cone, and the universally pseudoeffective cone. The smallest of these; the Pliant cone $\mathrm{Pl}^m(\mathrm{X}) \subset \mathrm{Nef}^m(\mathrm{X})$ is the closure of the cone generated by monomials in Schur classes of globally generated vector bundles on X.

2.4.2 A subcone of $\mathrm{Pl}^m(\overline{\mathrm{M}}_{0,n})$ from Vector bundles of conformal blocks

Claim 2.4.1. *There is a spanning set for $\mathrm{A}^m(\overline{\mathrm{M}}_{0,n})$, given by a basis of first Chern classes of vector bundles of conformal blocks. In particular, all classes lie in the pliant cone.*

Proof. By [Kee92], $A^1(\overline{\mathrm{M}}_{0,n-1})$ generates $A^k(\overline{\mathrm{M}}_{0,n-1})$, all k.

There is at least one basis we may use for the Picard group of $\overline{\mathrm{M}}_{0,n}$. Namely, the bundles $\mathcal{B}$ that generate Fakhruddin's basis for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})$, are

$$\mathcal{B} = \{\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}, 1) : \mathrm{rk}(\mathbb{V}(\mathfrak{sl}(2), \vec{\lambda}, 1) \neq 0\}.$$

In $\mathcal{B}$ bundles are determined by n-tuples of weights of the form $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \{0, \omega_1\}$, $0 \neq \sum_i |\lambda_i|$ is divisible by 2 and such that at least four weights λ_i are different than zero. Moreover, all have level one, and so also have rank one. We note that if n is odd, then all elements of $\mathcal{B}$ are pulled back from $\overline{\mathrm{M}}_{0,n-1}$ and if n is even, then $\mathbb{V}(\mathfrak{sl}(2), \{\omega_1^n\}, 1)$ is the unique element of $\mathcal{B}$ that is not pulled back from $\overline{\mathrm{M}}_{0,n-1}$.

□

Remark 2.4.2. *Swinarski showed that Fakhruddin's basis does not cover the whole nef cone of* $\overline{\mathrm{M}}_{0,6}$*, and so it isn't likely that these could be used to show that the cones spanned by conformal blocks classes and nef cones are the same for* $k > 1$*.*

Lecture 3

Appendices

3.1 What is a moduli space, technically speaking?

3.1.1 The functor of points

Definition 3.1.1. *Let* X *be a scheme over a field k. The* ***functor of points*** *of a scheme* X *is the contravariant functor*

$$h_X : (\mathcal{S}ch_k) \to (\mathcal{S}ets),$$

from the category $(\mathcal{S}ch_k)$ *of schemes over k to the category* $(\mathcal{S}ets)$ *of sets which takes a scheme* $Y \in \mathrm{Ob}(\mathcal{S}ch_k)$ *to the set* $h_X(Y) = \mathcal{M}or_{\mathcal{S}ch_k}(Y, X)$*, and takes maps of schemes* $f : Y \to Z$*, to maps of sets:*

$$h_X(f) : h_X(Z) \to h_X(Y), \ \ [g : Z \to X] \mapsto [g \circ f : Y \to X].$$

Definition 3.1.2. *We say that a contravariant functor*

$$F : (\mathcal{S}ch_k) \to (\mathcal{S}ets),$$

is representable *if it is of the form* h_X *for some scheme* X*. By Yoneda's Lemma (below), if* X *exists, then it is unique, and we say that* X *represents the functor F.*

For a proof of **Yoneda's Lemma**, which we next state, see for example [EH00, pages 252-253]

Lemma 3.1.3 (Yoneda)**.** *Let* C *be a category and* X*, and let* $X' \in \mathrm{Obj}(C)$*.*

(a) If F is any contravariant functor from $\mathcal{C}$ to the category of sets, the natural transformations from $\mathcal{M}or(\ ,\mathrm{X})$ to F are in natural correspondence with the elements of $F(\mathrm{X})$;

(b) If functors $\mathcal{M}or(\ ,\mathrm{X})$ and $\mathcal{M}or(\ ,\mathrm{X}')$ are isomorphic, then $\mathrm{X} \cong \mathrm{X}'$.

3.1.2 Fine moduli spaces

See also [Kol96, Chapter 1], [EH00, Chapter VI, page], Kleiman's article on the Picard Scheme in [FGI+05], and [HM98].

Definition 3.1.4. *Given a reasonable[1] collection of objects $\mathcal{S}$, we define a (contravariant) moduli functor from the category $(\mathcal{S}ch_k)$ of schemes over k to the category $(\mathcal{S}ets)$ of sets*

$$\mathcal{F}_{\mathcal{S}} : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \mathcal{F}_{\mathcal{S}}(\mathrm{T}),$$

where $\mathcal{F}_{\mathcal{S}}(\mathrm{T})$ is equal to the set of flat families of objects in $\mathcal{S}$ parametrized by T up to isomorphism over T.

The question one then asks is whether there is a scheme which we can call $\mathrm{Mod}_{\mathcal{S}}$, or better said, a flat morphism of schemes:

$$u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}},$$

which is a fine moduli space for the moduli functor. This means that for every object $\mathrm{T} \in \mathrm{Obj}(\mathcal{S}ch_k)$, pulling back, gives an equivalence of sets:

$$\mathcal{F}_{\mathcal{S}}(\mathrm{T}) = \mathcal{M}or_{Sch}(\mathrm{T}, \mathrm{Mod}_{\mathcal{S}}).$$

For example, taking $\mathrm{T} = \mathrm{Mod}_{\mathcal{S}}$, we obtain the universal family $u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}}$ which corresponds to the identity element $\mathrm{id} \in \mathcal{M}or_{Sch}(\mathrm{Mod}_{\mathcal{S}}, \mathrm{Mod}_{\mathcal{S}})$. And taking $\mathrm{T} = \mathrm{Spec}(k)$, we see that the set of k-points of $\mathrm{Mod}_{\mathcal{S}}$ corresponds to the fibers of the family $u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}}$.

Another more formal way to say this is the following.

Definition 3.1.5. *The functor $\mathcal{F}_{\mathcal{S}}$ from Definition 3.1.4 is represented by the scheme $\mathrm{Mod}_{\mathcal{S}}$ if there is a natural isomorphism between $\mathcal{F}_{\mathcal{S}}$ and the functor of points $\mathcal{M}or_{Sch}(\quad, \mathrm{Mod}_{\mathcal{S}})$. In this case we say $\mathrm{Mod}_{\mathcal{S}}$ is **a fine moduli space** for the functor $\mathcal{F}_{\mathcal{S}}$.*

[1] As part of being a reasonable collection of objects, we require that $\mathcal{S}$ is closed under base extension. So for example, if objects X in $\mathcal{S}$ are defined over $\mathrm{Spec}(k)$, where k is a field, and if $k \hookrightarrow \mathrm{k}$ is a field extension, then $\mathrm{X}_{\mathrm{k}} = \mathrm{X} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathrm{k})$ is also in $\mathcal{S}$.

3.1.3 Example: The Grassmannian

Let S be a scheme of finite type over a field k, and let $(\mathcal{S}ch_{\mathrm{S}})$ denote the category of schemes of finite type over S. Fix two integers $0 < d < r$. We will consider the contravariant functor from $(\mathcal{S}ch_{\mathrm{S}})$ to the category $(\mathcal{S}ets)$ of sets:

$$\mathfrak{g}_{\mathrm{S}}^{r,d} : (\mathcal{S}ch_{\mathrm{S}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \mathfrak{g}_{\mathrm{S}}^{r,d}(\mathrm{T}),$$

such that

$$\mathfrak{g}_{\mathrm{S}}^{r,d}(\mathrm{T}) = \{\, q : \mathcal{O}_{\mathrm{T}}^{r} \twoheadrightarrow \mathcal{F} : \ \mathcal{F} \ \text{ a coherent locally free } \mathcal{O}_{\mathrm{T}} \text{ -module of rank } d \,\}/\sim,$$

where two quotients $q_1 : \mathcal{O}_{\mathrm{T}}^{r} \twoheadrightarrow \mathcal{F}_1$ and $q_2 : \mathcal{O}_{\mathrm{T}}^{r} \twoheadrightarrow \mathcal{F}_2$ in $\mathfrak{g}_{\mathrm{S}}^{r,d}(\mathrm{T})$ are equivalent if there is an isomorphism $f : \mathcal{F}_1 \to \mathcal{F}_2$, making the diagram

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{f} & \mathcal{F}_1 \\ & {\scriptstyle q_1}\nwarrow & \uparrow{\scriptstyle q_2} \\ & & \mathcal{O}_T^r \end{array}$$

commute. Grothendieck proved that there is a projective scheme $\mathrm{G}_{\mathrm{S}}^{r,d}$ of finite type over S (ie an object in $(\mathcal{S}ch_{\mathrm{S}})$) that represents the functor $\mathfrak{g}_{\mathrm{S}}^{r,d}$.

One can generalize the Grassmannian, forming Hilbert schemes, and Quot schemes for example.

3.1.4 Example: Hilbert schemes

If X is a projective scheme of finite type over S, we can consider the contravariant functor

$$h_{\mathrm{X/S}} : (\mathcal{S}ch_{\mathrm{S}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto h_{\mathrm{X/S}}(\mathrm{T}),$$

and for $\mathrm{X}_T = \mathrm{X} \times_{\mathrm{S}} \mathrm{T}$, one has $h_{\mathrm{X/S}}(\mathrm{T}) = \{q : \mathcal{O}_{\mathrm{X_T}} \twoheadrightarrow \mathcal{F} : \ \mathcal{F} \text{ satisfying (1) and (2)}\}/\sim$.

(a) is a coherent sheaf of $\mathcal{O}_{\mathrm{T}}$-modules; and

(b) is flat and with compact support with respect to the projection $p_2 : \mathrm{X} \times_{\mathrm{S}} \mathrm{T} \to \mathrm{T}$.

Notice here that $r = 1$, and as $\mathcal{O}_{\mathrm{X_T}}$ is the ring of regular functions for $\mathrm{X}_T = \mathrm{X} \times_{\mathrm{S}} \mathrm{T}$, by taking kernels of the maps $q : \mathcal{O}_{\mathrm{X_T}} \twoheadrightarrow \mathcal{F}$, we get that the set $h_{\mathrm{X/S}}(\mathrm{T})$ is in bijection with the set of closed subschemes of X parametrized by T. Grothendieck showed this functor is representable by the Hilbert scheme $\mathrm{Hilb}_{\mathrm{X/S}}$, which while not of finite type over S, is a union of schemes of finite type, parametrized by Hilbert polynomials, each of which represents a moduli functor. We'll speak more about these.

3.1.5 Example: Quot schemes

A common generalization of the previous two examples are the following two contravariant functors.

$$\mathcal{Q}_{\mathcal{O}^r_X/X/S} : (\mathcal{S}ch_S) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{Q}_{\mathcal{O}^r_X/X/S}(T),$$

such that, for $X_T = X \times_S T$, the set $\mathcal{Q}_{\mathcal{O}^r_X/X/S}(T)$ is equal to

$$\{q : \mathcal{O}^r_{X_T} \twoheadrightarrow \mathcal{F} : \mathcal{F} \text{ coherent } \mathcal{O}_{X_T}\text{-module, flat with compact support over } T\}/\sim .$$

More generally, if $\mathcal{E}$ is a locally free sheaf on X, we define a contravariant functor

$$\mathcal{Q}_{\mathcal{E}/X/S} : (\mathcal{S}ch_S) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{Q}_{\mathcal{E}/X/S}(T),$$

where for $p_1 : X_T = X \times_S T \to X$ the projection onto the first factor, the set $\mathcal{Q}_{\mathcal{E}/X/S}(T)$ is

$$\{q : p_1^*\mathcal{E} \twoheadrightarrow \mathcal{F} : \mathcal{F} \text{ coherent } \mathcal{O}_{X_T}\text{-module, flat with compact support over } T\}/\sim .$$

Grothendieck proved that $\mathcal{Q}_{\mathcal{E}/X/S}$ is represented by the so-called Quot-scheme $\mathrm{Quot}_{\mathcal{E}/X/S}$, which while not finite type over S, again is a union of schemes of finite type over S, parametrized by Hilbert polynomials.

3.1.6 Not an example: the moduli space of smooth curves

Consider, for $g = \dim H^1(C, \mathcal{O}) \geqslant 2$:

$$\mathcal{M}_g : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{M}_g(T),$$

where $\mathcal{M}_g(T)$ is the set of proper flat maps $\pi : \mathcal{F} \to T$ such that every fiber $\mathcal{F}_t$ is a smooth projective curve of genus g modulo isomorphism over T. This functor is not represented by a fine moduli space: every curve with nontrivial automorphisms creates issues.

Example 3.1.6. *We will consider a nontrivial family of hyperelliptic curves parametrized by $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$. To describe this family, let $X = Z(y^2 - f(x))$ be any smooth hyperelliptic curve of genus g with $\mathrm{Aut}(X) \cong C_2 =< \tau >$. The cyclic group C_2 acts on X and on $\mathbb{G}_m$:*

$$C_2 \times X \to X, \ (\tau, (x, y)) \mapsto (x, -y), \quad \textit{and} \quad C_2 \times \mathbb{G}_m \to \mathbb{G}_m, \ (\tau, z) \mapsto -z;$$

and we can form the contracted product

$$\mathcal{F} = \mathbb{G}_m \times_{C_2} X = (\mathbb{G}_m \times X)/\sim, \quad \textit{where} \quad (\tau \cdot \alpha, p) \sim (\alpha, \tau \cdot p).$$

We'll set

$$\pi : \mathcal{F} \to \mathbb{G}_m \;\; [(\alpha, p)] \mapsto \alpha^2,$$

which is well defined since by this prescription $(\tau \cdot \alpha, p) = (-\alpha, p) \mapsto \alpha^2$, *and* $(\alpha, \tau \cdot p) \mapsto \alpha^2$. *To see that fibers of* π *are isomorphic to* X, *notice that one can view the set of points lying over* $\alpha^2 \in \mathbb{G}_m$ *as all points lying on two copies of* X *that are identified by the equivalence relation* $\sim$. *In particular if the functor* $\mathcal{M}_g$ *were represented by a fine moduli space* M_g *with a universal family* $u : U_g \to M_g$, *then there would be a constant map*

$$\mu_\pi : \mathbb{G}_m \to M_g, \;\; \alpha \mapsto [X],$$

and so $\mathcal{F}$ *would be equal to the constant family, giving a commutative diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{F} & \mathbb{G}_m \times X \\ & \searrow^{\pi} & \downarrow p_1 \\ & & \mathbb{G}_m. \end{array}$$

But the map $F : \mathcal{F} \to \mathbb{G}_m \times X$ *could simply not be well defined, for all points* $[(\alpha, p)] \in \mathcal{F}$, *and so this is impossible.*

However, there is a scheme M_g with the following properties:

(a) for an algebraically closed field k, the k-points of M_g are in one to one correspondence with the set of isomorphism classes of smooth curves of genus g defined over k;

(b) if $\pi : \mathcal{F} \to T$ is a flat family of curves of genus g, then there is a map $\mu_\pi : T \to M_g$ such that if $t \in T$ is a geometric point, then $\mu_\pi(t)$ is the point $[\mathcal{F}_t]$ in M_g corresponding to the isomorphism class of the fiber $\mathcal{F}_t = \pi^{-1}(t)$.

3.1.7 Coarse moduli spaces

Definition 3.1.7. *We say that a scheme* $\mathrm{Mod}_{\mathcal{S}}$ *is a* ***coarse moduli space*** *for the functor* $\mathcal{F}_{\mathcal{S}}$ *(from Definition 3.1.4), if*

(a) there is a natural transformation of functors $\mathcal{F}_{\mathcal{S}} \to \mathcal{M}or_{Sch}(\;\;, \mathrm{Mod}_{\mathcal{S}})$*;*

(b) the scheme $\mathrm{Mod}_{\mathcal{S}}$ *is universal for* (1)*;*

(c) for any algebraically closed field extension $\mathrm{k} \hookrightarrow \mathrm{K}$*,*

$$\mathcal{F}_{\mathcal{S}}(\mathrm{K}) \cong \mathcal{M}or_{\mathcal{S}ch}(\mathrm{Spec}(\mathrm{K}), \mathrm{Mod}_{\mathcal{S}}) = \mathrm{Mod}_{\mathcal{S}}(\mathrm{K}),$$

is an isomorphism of sets.

In particular, the scheme M_g is a coarse moduli space for the functor $\mathcal{M}_g$ described in Section 3.1.6.

Definition 3.1.8. *For* $g = \dim \mathrm{H}^1(C, \mathcal{O}_C) \geqslant 2$*, consider the contravariant functor:*

$$\overline{\mathcal{M}}_g : (\mathcal{S}ch_{\mathrm{k}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \overline{\mathcal{M}}_g(\mathrm{T}),$$

where $\overline{\mathcal{M}}_g(\mathrm{T})$ *is the set of flat proper morphisms* $\pi : \mathcal{F} \to \mathrm{T}$ *such that every fiber* $\mathcal{F}_t$ *is a stable curve of genus* g *modulo isomorphism over* T*.*

Theorem 3.1.9. [DM69] *There exists a coarse moduli space* $\overline{\mathrm{M}}_g$ *for the moduli functor* $\overline{\mathcal{M}}_g$*; Moreover,* $\overline{\mathrm{M}}_g$ *is a projective variety that contains* M_g *as a dense open subset.*

Remark 3.1.10. *Let* T *be any smooth curve and* $p \in \mathrm{T}$ *a (geometric) point on* T*. Suppose there is a regular map*

$$\mu^* : \mathrm{T}^* = \mathrm{T} \backslash \{p\} \to \overline{\mathrm{M}}_g.$$

By definition of coarse moduli space, this map corresponds to a family $\pi : \mathrm{X} \to \mathrm{T}^*$ *of stable curves of genus* g*, parametrized by* T^**. Now by Theorem 3.1.9, the moduli space* $\overline{\mathrm{M}}_g$ *is proper, and so by the valuative criterion for properness, there is an extension of* μ^* *giving a morphism* $\mu : \mathrm{T} \to \overline{\mathrm{M}}_g$*. But by Theorem 3.1.9,* $\overline{\mathrm{M}}_g$ *is also separated, and one can use this to show this extension* μ *is unique. So this says that there is a unique extension to a family* $\pi : \mathrm{X} \to \mathrm{T}$ *parametrized by* T*. This is the content of* ***the stable reduction theorem****.*

3.2 Generalizations based on the combinatorial structure of the boundary

3.2.1 Tropical curves and moduli of tropical curves

A tropical curve is a metric weighted graph: A pair $G = (\Gamma, \ell : E(\Gamma) \to \mathbb{R}_{>0} \cup \{\infty\})$ consisting of a weighted graph and a length function assigned to each edge. Fixing a weighted graph Γ, the tropical curves having graph isomorphic to Γ are determined by the lengths of their edges. One can define:

$$\overline{\mathrm{M}}_{\Gamma}^{\mathrm{Trop}} := (\mathbb{R}_{>0} \cup \{\infty\})^{|E(\Gamma)|} / \operatorname{Aut}(\Gamma),$$

and show that these glue together to form a moduli space:

$$\overline{\mathrm{M}}_{g}^{\mathrm{Trop}} = \cup_{g(\Gamma)=g} \overline{\mathrm{M}}_{\Gamma}^{\mathrm{Trop}},$$

that Sam will talk about Moduli of tropical curves and the relationship between moduli of curves and moduli of tropical curves [Abr13] describing [ACP15]: [BPR16, Tyo12, Viv13, ACP15, Ber90, Thu07, KKMSD73]

3.3 How can studying $\overline{\mathrm{M}}_g$ tell us about curves?

We started by considering a family of curves parametrized by an open subset of $\mathbb{A}^6$, that included the general smooth curve of genus 2.

Generally speaking, if there is a family of curves parametrized by an open subset of A^{N+1} that includes the general curve of genus g, then one would have a dominant rational map from $\mathbb{P}^N$ to our compactification $\overline{\mathrm{M}}_g$. In other words, $\overline{\mathrm{M}}_g$ would be unirational. This would imply that there are no pluricanonical forms on $\overline{\mathrm{M}}_g$. Said otherwise still, the canonical divisor of $\overline{\mathrm{M}}_g$ would not be effective.

On the other hand, one of the most important results about the moduli space of curves, proved almost 40 years ago, is that for $g >> 0$ the canonical divisor of $\overline{\mathrm{M}}_g$ lives in the interior of the cone of effective divisors (for $g = 22$ and $g \geqslant 24$, by [EH87, HM82], and for by $g = 23$ [Far00]). Once the hard work was done to write down the classes of the canonical divisor, and an effective divisor called the Brill-Noether locus, to prove this famous result, a very easy combinatorial argument can be made to show that the canonical divisor is equal to an effective linear combination of the Brill-Noether and boundary divisors when the genus is large enough.

The upshot is that by shifting focus to the geometry of the moduli space of curves, we learn something basic and valuable about the existence of equations of smooth general curves. Moreover, for these values of g for which $\overline{\mathrm{M}}_g$ is known to be of general type, one can consider the canonical ring

$$\mathrm{R}_{\bullet} = \bigoplus_{m \geqslant 0} \Gamma(\overline{\mathrm{M}}_g, m\,\mathrm{K}_{\overline{\mathrm{M}}_g}),$$

which is now known to be finitely generated by the celebrated work of [BCHM10]. In particular, the canonical model $\mathrm{Proj}(R_\bullet)$, is birational to $\overline{M}_g$.

It is still an open problem to construct this model, and efforts to achieve this goal have both furthered our understanding of the birational geometry of the moduli space of curves, as well as giving a highly nontrivial example where this developing theory can be experimented with and better understood.

3.3.1 Chow rings for general proper varieties using Chern classes of vector bundles

Definition 3.3.1. *Let $A_k(X)$ be the group of algebraic cycles of dimension k on X.*

In his book on Intersection theory, Fulton defines a Chern class as a linear operator:

Definition 3.3.2. *Let X be a proper variety, and $\mathcal{E}$ a vector bundle on X. The r-th Chern class of $\mathcal{E}$ is a linear operator*

$$c_r(\mathcal{E}) : A_k(X) \to A_{k-r}(X).$$

Definition 3.3.3. *Two cycles Z_1 and Z_2 on X are numerically equivalent if for every weight k monomial p in Chern classes of vector bundles, one has*

$$\deg(P \cdot Z_1) = \deg(P \cdot Z_2).$$

This defines a pairing between weight k-Chern classes and cycles of dimension k.

Definition 3.3.4. $N_k(X)_{\mathbb{Z}} = A_k(X)/$ *numerical equivalence .*

Definition 3.3.5. *The finitely generated Abelian group $N_k(X)_{\mathbb{Z}}$ is a lattice in the vector space $N_k(X) = N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$.*

Definition 3.3.6. *The pseudo effective cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is defined to be the closure of the cone generated by cycles with nonnegative coefficients.*

The cone $\overline{\mathrm{Eff}}_k(X)$ is full dimensional, spanning the vector space $N_k(X)$. It is pointed (containing no lines), closed, and convex.

Definition 3.3.7. *Its dual of the vector space $N_k(X)$ is:*

$$N^k(X) = \{\mathbb{R} \text{ polynomials in weight } k\text{-Chern classes }\}/\equiv,$$

where equivalence $\equiv$ is given by intersection with cycles.

Definition 3.3.8. *The Nef Cone* $\mathrm{Nef}^k(X) \subset \mathrm{N}^k(X)$ *is the cone dual to* $\overline{\mathrm{Eff}}_k(X)$*.*

As the dual of $\overline{\mathrm{Eff}}_k(X)$, the nef cone has all of the nice properties that $\overline{\mathrm{Eff}}_k(X)$ does.

Example 3.3.9. *By the definition given above,* $\mathrm{N}^1(X) = \{$ *first Chern classes* $\}/\equiv$*, where* $\equiv$ *is defined by intersection with* 1*-cycles. This is the same as what you are used to seeing because if* $\mathcal{E}$ *is any vector bundle, then* $c_1(\mathcal{E}) = c_1(\det(\mathcal{E}))$*, and* $\det(\mathcal{E})$ *is a line bundle.*

3.4 Just enough about affine Lie algebras

3.4.1 $\hat{\mathfrak{g}} = \big(\mathfrak{g} \otimes k((\xi))\big) \oplus \mathbb{C} \cdot c$

Let $\mathfrak{g}$ be a Lie algebra with bracket $[\ ,\ ]$. In this section we will define and study the affine Lie algebra

$$\hat{\mathfrak{g}} = \big(\mathfrak{g} \otimes \mathbb{C}((\xi))\big) \oplus \mathbb{C} \cdot c,$$

where $\mathbb{C}((\xi))$ is the field of Laurant power series over $\mathbb{C}$ in 1 variable, and $c \in \mathfrak{g}$ is in the center of $\hat{\mathfrak{g}}$. To define the bracket for $\hat{\mathfrak{g}}$, we set

$$[\mathrm{X} \otimes f(\xi), \mathrm{Y} \otimes g(\xi)] = [\mathrm{X}, \mathrm{Y}] \otimes f(\xi) g(\xi) + (\mathrm{X}, \mathrm{Y}) \cdot \mathrm{Res}(g(\xi) d f(\xi)) \cdot c,$$

where $\mathrm{X}, \mathrm{Y} \in \mathfrak{g}$.

Typical elements in $\hat{\mathfrak{g}}$ are of the form $\sum_{i=1}^n \mathrm{X}_i \otimes f_i(\xi) \oplus \lambda c$, and $\sum_{j=1}^n \mathrm{Y}_j \otimes g_j(\xi) \oplus \mu c$, so using that $[c, \square] = [\square, c] = 0$, for all $\square \in \hat{\mathfrak{g}}$, since c is central in $\hat{\mathfrak{g}}$:

$$\begin{aligned}(3.1)\quad & [\sum_{i=1}^n \mathrm{X}_i \otimes f_i(\xi) \oplus \lambda c, \sum_{j=1}^n \mathrm{Y}_j \otimes g_j(\xi) \oplus \mu c] \\ &= [\sum_{i=1}^n \mathrm{X}_i \otimes f_i(\xi), \sum_{j=1}^n \mathrm{Y}_j \otimes g_j(\xi)] = \sum_{ij} [\mathrm{X}_i \otimes f_i(\xi), \mathrm{Y}_j \otimes g_j(\xi)].\end{aligned}$$

So the upshot is that we really only need to know that the given definition for $[\mathrm{X} \otimes f(\xi), \mathrm{Y} \otimes g(\xi)]$ makes sense and is well defined. That is, we need to check anti-symmetry and the Jacobi identity.

Claim 3.4.1. *The proposed Lie bracket for* $\hat{\mathfrak{g}}$ *satisfies the Jacobi identity:*

$$\begin{aligned}(3.2)\quad & [[\mathrm{X} \otimes f(\xi), \mathrm{Y} \otimes g(\xi)], \mathrm{Z} \otimes h(\xi)] \\ &= [\mathrm{X} \otimes f(\xi), [\mathrm{Y} \otimes g(\xi), \mathrm{Z} \otimes h(\xi)]] - [\mathrm{Y} \otimes g(\xi), [\mathrm{X} \otimes f(\xi), \mathrm{Z} \otimes h(\xi)]].\end{aligned}$$

Proof. Using a bit of shorthand, we drop the variable ξ writing $\overline{gf'}$ instead of $\mathrm{Res}(g(\xi)df(\xi))$, we can express the left hand side of the equation as:

$$[[X,Y]\otimes fg \oplus \big((X,Y)\overline{gf'}\big)c, Z\otimes h] = A + B, \tag{3.3}$$

where

$$A = [[X,Y],Z]\otimes fgh, \quad \text{and} \quad B = \big([X,Y],Z\big)\overline{h(fg)'}c.$$

The right hand side of the equation can be written as:

$$\begin{aligned}
&[X\otimes f, [Y,Z]\otimes gh + (Y,Z)\overline{g'h}\cdot c] - [Y\otimes g, [X,Z]\otimes fh + (x,Z)\overline{hf'}c] \\
&\qquad = [X,[Y,Z]]\otimes fgh \oplus (X,[Y,Z])\cdot \overline{ghf'}\cdot c \\
&\qquad\qquad \ominus [Y,[X,Z]]\otimes gfh \oplus (Y,[X,Z])\cdot \overline{fhg'}\cdot c = A' + B',
\end{aligned} \tag{3.4}$$

where

$$A' = [X,[Y,Z]]\otimes fgh, \quad \text{and} \quad B' = \Big(\big(X,[Y,Z]\big)\overline{ghf'} - \big(Y,[X,Z]\big)\overline{fhg'}\Big)c.$$

One has that $A = A'$ by the Jacobi identity for the Lie bracket for $\mathfrak{g}$, and so it remains to check that $B = B'$. Using the following three identities:

(a) the product rule: $(fg)' = f'g + fg'$;

(b) $\big([X,Y],Z\big) = \big(X,[Y,Z]\big)$ (Lemma 3.4.2); and

(c) $\big([X,Y],Z\big) = -\big(Y,[X,Z]\big)$ (Lemma 3.4.3),

we write

$$\begin{aligned}
B &= \big([X,Y],Z\big)\overline{h(fg)'c} \\
&= \big([X,Y],Z\big)\overline{hf'gc} + \big([X,Y],Z\big)\overline{hfg'c} \\
&= \big(X,[Y,Z]\big)\overline{hf'gc} - \big(Y,[X,Z]\big)\overline{hfg'c} = B'.
\end{aligned} \tag{3.5}$$

□

The following identity is referred to as the Frobeneous property of the Killing form.

Lemma 3.4.2. $\big([X,Y],Z\big) = \big(X,[Y,Z]\big)$

Proof. By definition of the Killing form, and using that Trace is invariant under cyclic permutations (so Trace(abc) = Trace(cab)), we write:

$$(3.6)\quad \big(X,[Y,Z]\big) = \text{Trace}\big(\text{ad}(X)\,\text{ad}(Y)\,\text{ad}(Z)\big) - \text{Trace}\big(\text{ad}(X)\,\text{ad}(Z)\,\text{ad}(Y)\big)$$
$$= \text{Trace}\big(\text{ad}(X)\,\text{ad}(Y)\,\text{ad}(Z)\big) - \text{Trace}\big(\text{ad}(Y)\,\text{ad}(X)\,\text{ad}(Z)\big)$$
$$\text{Trace}\big((\text{ad}(X)\,\text{ad}(Y) - \text{ad}(Y)\,\text{ad}(X)\,\text{ad}(Z)\big) = \big([X,Y],Z\big).$$

□

Lemma 3.4.3. $\big([X,Y],Z\big) = -\big(Y,[X,Z]\big)$

Proof. For the left hand side of the equation, using the symmetry of the Killing form:

$$\big([X,Y],Z\big) = \big(Z,[X,Y]\big).$$

For the right hand side, using that the Lie bracket is antisymmetric, while the Killing form is symmetric, we write:

$$-\big(Y,[X,Z]\big) = \big(Y,[Z,X]\big) = \big([Z,X],Y\big).$$

Now these are the same by Lemma 3.4.2:

$$\big([Z,X],Y\big) = \big(Z,[X,Y]\big).$$

□

Claim 3.4.4. $[X\otimes f(\xi), X\otimes f(\xi)] = 0$

Proof. Using that $[X,X] = 0$ since $\mathfrak{g}$ is a Lie algebra, and it's Lie bracket is of course anti-symmetric, and moreover, since $\frac{d}{d\xi}\frac{1}{2}f^2 = f(\xi)f'(\xi)d\xi$. So

$$\text{Res}_{\xi=0}\big(\frac{d}{d\xi}\frac{1}{2}f^2\big) = \text{Res}_{\xi=0}\big(f(\xi)f'(\xi)d\xi\big) = 0.$$

We can then write

$$[X\otimes f(\xi), X\otimes f(\xi)] = \big([X,X]\otimes f(\xi)\cdot f(\xi)\big) \oplus (x,x)\,\text{Res}_{\xi=0}\big(f(\xi)f'(\xi)d\xi\big) = 0.$$

□

The universal enveloping algebra

From any associative algebra A one can build a Lie algebra $\mathcal{L}(A)$ by taking the Lie bracket to be the commutator. Given a Lie algebra, we can also construct an associative algebra called the universal enveloping algebra – it has many of the features of the Lie algebra we start with but is in some sense easier to work with, and allows us to construct our vector bundle.

Definition 3.4.5. *For any (possibly infinite dimensional) Lie algebra $\mathfrak{g}$, the **universal enveloping algebra** of $\mathfrak{g}$ is defined to be any pair (U, i) where U is an associative algebra with unity and $i : \mathfrak{g} \to \mathcal{L}(U)$ is a homomorphism of Lie algebras with the property that, if A is any other associative algebra with unity and if $\phi : \mathfrak{g} \to \mathcal{L}(A)$ is any Lie algebra homomorphism, then there is a unique homomorphism of unital algebras $\psi : U \to A$, so that the following diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & \mathcal{L}(U) \\ & \underset{\phi}{\searrow} & \downarrow \psi_* \\ & & \mathcal{L}(A). \end{array}$$

commutes. In the diagram, the map ψ_* is equal to ψ, considered as a homomorphism of Lie algebras.

The universal enveloping algebra $(U(\mathfrak{g}), i)$ is constructed from the tensor algebra $\mathcal{T}(\mathfrak{g})$.

Definition 3.4.6. *Given a vector space V over a field k, the **tensor algebra** $\mathcal{T}(V)$ is defined to be the direct sum*

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} T^k(V), \text{ where } T^k(V) = V^{\otimes k} = V \otimes V \otimes \cdots \otimes V,$$

with multiplication determined by the canonical isomorphism

$$T^k(V) \bigotimes T^m(V) \to T^{k+m}(V),$$

given by the tensor product and extended linearly to all of $T(V)$.

Definition 3.4.7. *Let $\mathfrak{g}$ be a Lie algebra. Then set $U(\mathfrak{g})$ equal to the quotient of $\mathcal{T}(\mathfrak{g})$ by the ideal generated by all elements of the form*

$$\mathrm{X} \otimes \mathrm{Y} - \mathrm{Y} \otimes \mathrm{X} - [\mathrm{X}, \mathrm{Y}],$$

for all X *and* Y $\in \mathfrak{g}$, *and define*

$$i : \mathfrak{g} \to U(\mathfrak{g}), \;\; X \mapsto X\,.$$

Excercise 3.4.8. *Check that the relations defining* $U(\mathfrak{g})$ *ensure that* $i : \mathfrak{g} \to U(\mathfrak{g})$ *is a morphism of Lie algebras, and that* $(U(\mathfrak{g}), i)$ *is a universal enveloping algebra. Show that the universal enveloping algebra* (U, i) *of* $\mathfrak{g}$ *is unique up to isomorphism.*

3.4.2 $\hat{\mathfrak{g}}$ modules M_λ and H_λ

There is a bijection between the intersection $\mathcal{W} \cap \Lambda_W$ of the Weyl chamber $\mathcal{W}$ and the weight lattice Λ_W and the set of irreducible representations for a given Lie algebra $\mathfrak{g}$. Given $\lambda \in \mathcal{W} \cap \Lambda_W$, there is a corresponding finite irreducible representation V_λ for $\mathfrak{g}$. In particular, V_λ is a $\mathfrak{g}$-module.

We are going to use V_λ to construct a representation M_λ for $\hat{\mathfrak{g}}$.

To construct M_λ, we use the following:

$$\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi, \quad \text{and} \quad \hat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[[\xi^{-1}]]\xi^{-1},$$

which we regard as Lie subalgebras of $\hat{\mathfrak{g}}$. One can show that

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c \oplus \hat{\mathfrak{g}}_-.$$

We'll also use the "positive" Lie sub-algebra

$$\hat{\mathfrak{p}}_+ = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c,$$

along with the universal enveloping algebras $\mathcal{U}(\hat{\mathfrak{g}})$ and $\mathcal{U}(\hat{\mathfrak{p}}_+)$.

Definition 3.4.9. $M_\lambda := \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{p}}_+)} V_\lambda$.

Remark 3.4.10. *Definition 3.4.9 makes sense: taking such a tensor product is legal:*

(a) If $\mathfrak{g}_1$ *is any subalgebra of a Lie algebra* $\mathfrak{g}_2$*, then the inclusion* $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ *extends to a monomorphism* $\mathcal{U}(\mathfrak{g}_1) \hookrightarrow \mathcal{U}(\mathfrak{g}_2)$*. Furthermore* $\mathcal{U}(\mathfrak{g}_2)$ *is a free* $\mathcal{U}(\mathfrak{g}_1)$ *module. So in particular, as* $\hat{\mathfrak{p}}_+ \hookrightarrow \hat{\mathfrak{g}}$*, we have that* $\mathcal{U}(\hat{\mathfrak{g}})$ *is a free* $\mathcal{U}(\hat{\mathfrak{p}}_+)$ *module.*

(b) V_λ is a $\hat{\mathfrak{p}}_+$-module. To see that this is true, note that since V_λ is a $\mathfrak{g}$-representation, there is a Lie algebra homomorphism

$$\mathfrak{g} \to \mathrm{End}(V_\lambda).$$

Since $\hat{\mathfrak{p}}_+ = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c$, we can let $\hat{\mathfrak{g}}_+$ act by zero and $\mathbb{C} \cdot c$ act by taking

$$\mathbb{C} \cdot c \to \mathrm{End}(V_\lambda),\ \alpha c \mapsto [V_\lambda \to V_\lambda,\ \ v \mapsto (\alpha\ell)v],$$

where here ℓ is the level.

Claim 3.4.11. M_λ *is a representation for* $\hat{\mathfrak{g}}$

Proof. To show that there is a Lie algebra morphism

$$\hat{\mathfrak{g}} \to \mathrm{End}(\mathrm{M}_\lambda),$$

we may show there is a map of associative algebras

$$\mathcal{U}(\hat{\mathfrak{g}}) \to \mathrm{End}(\mathrm{M}_\lambda).$$

But by construction, $\mathcal{U}(\hat{\mathfrak{g}})$ acts on the left of M_λ, and so this is true. □

Definition 3.4.12. *We set* $\mathrm{H}_\lambda = \mathrm{M}_\lambda \,/\, \mathrm{I}_\lambda$*.*

Since $U(\hat{\mathfrak{g}})$ is isomorphic, as a $U(\hat{\mathfrak{g}}_-)$-module to $U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} U(\mathfrak{p}_+)$:

$$\begin{aligned} \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c = \mathfrak{g} \otimes \left(\mathbb{C}[[\xi^{-1}]]\xi^{-1} \otimes_{\mathbb{C}} \mathbb{C}[[\xi]]\xi\right) \oplus \mathbb{C} \cdot c \\ \cong \mathfrak{g} \otimes \mathbb{C}[[\xi^{-1}]]\xi^{-1} \otimes_{\mathbb{C}} \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi \oplus \mathbb{C} \cdot c \cong \hat{\mathfrak{g}}_- \otimes_{\mathbb{C}} \hat{\mathfrak{p}}_+. \end{aligned} \tag{3.7}$$

So we can rewrite the module M_λ as:

$$\mathrm{M}_\lambda \cong U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}}_+)} \mathrm{V}_\lambda \cong U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} U(\hat{\mathfrak{p}}_+) \otimes_{U(\hat{\mathfrak{p}}_+)} \mathrm{V}_\lambda \cong U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} \mathrm{V}_\lambda\,.$$

In particular, elements in $\mathrm{M}_\lambda := \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{p}}_+)} \mathrm{V}_\lambda$ look like elements $v \in \mathrm{V}_\lambda$ times all the negative stuff in $\hat{\mathfrak{g}}$.

With the notation above, M_λ contains a unique (see eg [TUY89, Bea96]) maximal proper submodule I_λ generated by an element

$$J_\lambda = (\mathrm{X}_\theta \otimes \xi_i^{-1})^{\ell-(\theta,\lambda)+1} \otimes v_\lambda \in \mathrm{M}_\lambda, \quad \text{and} \quad \mathrm{I}_\lambda = U(\hat{\mathfrak{g}}_-)J_\lambda,$$

where here θ is the longest root, $\mathrm{X}_\theta \in \mathfrak{g}$ is the corresponding coroot, and v_λ is the highest weight vector associated to λ. We set

$$\mathrm{H}_\lambda = \mathrm{M}_\lambda \,/\, \mathrm{I}_\lambda\,.$$

We see that H_λ is a $(\mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c)$-module. The subspace of H_λ annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic as a $\mathfrak{g}$-module to V_λ. So we identify V_λ with this subspace of H_λ annihilated by $\hat{\mathfrak{g}}_+$.

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