A combinatorial problem about vector bundles of conformal blocks on $\overline{M}_{0,n}$

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The problem I discuss today came up in joint work with P. Belkale, and S. Mukhopadhyay. It can be stated more generally, but I'll focus today on the case of conformal blocks for $\mathfrak{sl}_{r+1}$.

The moduli space of curves

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$\overline{\mathrm{M}}_{0,n}$ is a smooth projective variety, whose points correspond to stable n-pointed rational curves.

A stable n-pointed rational curve $(C, \vec{p})$ is:

- A rational curve C that may have (at worst) simple nodal singularities;
- $\vec{p} = (p_1, \ldots, p_n)$ are n smooth points on C; and
- The (n+1)-tuple $(C, \vec{p})$ has finitely many automorphisms.

Vector bundles of conformal blocks

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for $\mathfrak{sl}_{r+1}$ are given by:

(1) $\mathfrak{sl}_{r+1}$
(2) a positive integer ℓ;
(3) $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = \sum_{j=1}^{r} c_j \omega_j$, $c_j \geq 0, \forall j$, and $\sum_{j=1}^{r} c_j \leq \ell$.

such that

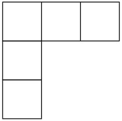
$$(r+1) | \sum_{i=1}^{n} |\lambda_i|,$$

where $|\lambda_i| = \sum_{j=1}^{r} j \cdot c_j$.

$V(\mathfrak{sl}_4, \{\omega_1, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3\}, 3)$

Weights correspond to Young diagrams:

$\lambda_1 = \omega_1$ 

$\lambda_2 = \lambda_3 = \lambda_4 = 2\omega_1 + \omega_3$ 

$$\sum_{i=1}^{n} |\lambda_i| = 1 + 3 \cdot 5 = 16.$$

For $g = 0$ the bundles are globally generated and so their first Chern classes, the conformal blocks divisors $c_1(\mathbb{V})$ are base point free, give morphisms.

Many symmetries and identities govern aspects of these bundles and divisors. Rank of the bundles plays a key role.

Goals are to

- Find order in the set of all conformal blocks divisors;
- Understand their associated maps.

Example: Additive Identities

Theorem (BGM)

Given $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\mu}, \ell)$*, of rank* 1*, and* $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\nu}, m)$*, such that* $\operatorname{rk} \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\mu}, \ell) = \operatorname{rk} \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\mu} + \vec{\nu}, \ell + m) = R$*,*

$$
\begin{aligned}
&c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\mu} + \vec{\nu}, \ell + m)) \\
&\qquad = R c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\mu}, \ell)) + c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\nu}, m)).
\end{aligned}
$$

Specific example

The following bundles have rank one:

- $V(\mathfrak{sl}_4, \omega_1^4, 1))$;
- $V(\mathfrak{sl}_4, \{0, (\omega_1 + \omega_3)^3\}, 2))$;
- $V(\mathfrak{sl}_4, \{\omega_1, (2\omega_1 + \omega_3)^3\}, 3)$

so

$$c_1 V(\mathfrak{sl}_4, \{\omega_1, (2\omega_1 + \omega_3)^3\}, 3) = c_1 V(\mathfrak{sl}_4, \omega_1^4, 1) + c_1 V(\mathfrak{sl}_4, \{0, (\omega_1 + \omega_3)^3\}, 2)). \quad (1)$$

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Two (equivalent) ways to say this:

- Given $\mathbb{V}$, is $c_1(\mathbb{V}) \neq 0$?
- Given $\mathbb{V}$, can one find a curve C such that

$$c_1(\mathbb{V}) \cdot C \neq 0?$$

Theorem (BGM)

$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = 0$ *if either*

1. $\ell > -1 + \frac{1}{r+1} \sum_{i=1} |\lambda_i| = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$*; or if*
2. $\ell > -1 + \frac{1}{2} \sum_{i=1} \lambda_i^{(1)} = \theta(\mathfrak{sl}_{r+1}, \vec{\lambda})$.

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For instance,

$$cl(\mathfrak{sl}_4, \omega_1^4) = 0, \quad \theta(\mathfrak{sl}_4, \omega_1^4) = 1,$$

hence $1 > cl(\mathfrak{sl}_4, \omega_1^4)$, and so

$$c_1\mathbb{V}(\mathfrak{sl}_4, \omega_1^4, 1) = 0.$$

Theorem (BGM)

$c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ is nontrivial if and only if $\operatorname{rk} \mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) > 0$ *and $1 \leq \ell \leq cl(\mathfrak{sl}_2, \vec{\lambda}) = \theta(\mathfrak{sl}_2, \vec{\lambda})$.*

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But not so simple in general:

$$c_1(\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3\}, 3)) = 0,$$

even though the level is below the theta and critical level, and the rank of the bundle itself is one.

When is the first Chern class nontrivial?

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = \sum_{i=1}^{n} c(\lambda_i)\,\mathrm{rk}\,\mathbb{V}(\vec{\lambda})\ \psi_i$$

$$- \sum_{I \subset [n]} \Big(\sum_{\mu} c(\mu)\,\mathrm{rk}\,\mathbb{V}(\vec{\lambda}(I) \cup \mu)\,\mathrm{rk}\,\mathbb{V}(\vec{\lambda}(I^C) \cup \mu*) \Big) \delta_I.$$

For the coefficients of the boundary classes, we sum over representations μ, such that

$$\operatorname{rk} \mathbb{V}(\vec{\lambda}(I) \cup \mu) = \operatorname{rk} \mathbb{V}(\mathfrak{sl}_{r+1}, \{\lambda_j : j \in I\} \cup \mu, \ell) > 0.$$

$$\operatorname{rk} \mathbb{V}(\vec{\lambda}(I^C) \cup \mu *) = \operatorname{rk} \mathbb{V}(\mathfrak{sl}_{r+1}, \{\lambda_j : j \in I^C\} \cup \mu^*, \ell) > 0.$$

For $\lambda = \sum_{i=1^r} c_i \omega_i$,

$$c(\lambda) = \frac{1}{r+1} \sum_{i=1}^{r} (r+1-i) i c_i^2 + \frac{1}{r+1} \sum_{1 \leq i < j \leq r} 2(r+1-j) i c_i c_j + \sum_{i=1}^{r} (r+1-i) i c_i. \quad (2)$$

Witten's Dictionary to compute $R = \text{rk}\,\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$

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- If $s > 0$, then let $\lambda = \ell\omega_1$. R is the coefficient of $q^s \sigma_{\ell\omega_{r+1}}$ in the quantum product

$$\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \star \sigma_{\lambda}^{s} \in \text{QH}^*(\text{Gr}(r+1, r+1+\ell)).$$

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- If $s \le 0$, then R is the multiplicity of the class of $\sigma_{k\omega_{r+1}}$ in the product

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \cdots \cdot \sigma_{\lambda_n} \in H^*(Gr(r+1, r+1+\ell+s)).$$

See (Theorem 3.6, Eq (3.10) and Remark 3.8) in Belkale's paper Quantum generalization of the Horn conjecture for a proof of Witten's dictionary.

Restated in the rank 1 case in terms of intersecting with some F-Curve positively

Given positive integers r and ℓ, and representations λ_1,λ_2,$\ldots$, λ_n for SL($r+1$) at level ℓ, is there a partition $[n] = N_1 \cup N_2 \cup N_3 \cup N_4$ for which the following nonnegative number is strictly positive:

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$$\sum_{\vec{\mu}} \left(\mathrm{rk}\, \mathbb{V}_{\vec{\mu}} \sum_{i=1}^{4} c(\mu_i) - \sum_{\nu} \sum_{\{ijk\}=\{234\}} \mathrm{rk}\, \mathbb{V}_{\mu_1 \mu_i \nu}\, \mathrm{rk}\, \mathbb{V}_{\mu_j \mu_k \nu^*} \right)?$$

Here we sum over representations $\vec{\mu} = (\mu_1, \ldots, \mu_4)$, such that for each i

$$\mathrm{rk}\, \mathbb{V}_{\lambda(N_i) \cup \mu_i^*} = \mathrm{rk}\, \mathbb{V}(\mathfrak{sl}_{r+1}, \{\lambda_j \in N_i\} \cup \mu_i^*, \ell) > 0.$$

Thank you!

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Beauville Laszlo 1994;
Faltings 1994;
Kumar, Narasimhan, Ramanathan 1994;
Laszlo, Sorger 1997;
Pauly 1996.

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$$A_\bullet^{(C,\vec{p})} = \bigoplus_{m\in\mathbb{Z}} V(\mathfrak{sl}_{r+1}, \{m\lambda_1, .., m\lambda_n\}, m\ell)^*_{(C,\vec{p})}$$
$$\cong \bigoplus_{m\in\mathbb{Z}} \mathrm{H}^0(X_{(C,\vec{p})}, L_{(C,\vec{p})}^{\otimes m}), \quad (3)$$

where if $\lambda_i = \sum_{i=1} c_i\omega_i$, then

$$m\,\lambda_i = \sum_{i=1}^{r} m\,c_i\,\omega_i.$$

In particular,

$$\mathrm{Proj}(A_\bullet^{C,\vec{p}}) \cong X_{(C,\vec{p})}.$$

For instance:

If C is a smooth curve of genus 2, then

$$A_{C,\vec{p}} = \bigoplus_{m\in\mathbb{Z}} V(\mathfrak{sl}_2, m)|^*_{[C]} \cong \bigoplus_{m\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(m)).$$

In other words,

$$\mathrm{Proj}(A^{C,\vec{p}}_\bullet) \cong \mathbb{P}^3,$$

and we say that $V(\mathfrak{sl}_2, m)|_{[C]}$ has a geometric interpretation.

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Anna Kazanova will talk about this result and the fact that sometimes geometric extensions do hold at singular stable curves.

Theorem

(BG) For every $r \geq 1$ and every $[C] \in \overline{\mathrm{M}}_g$, there is an ℓ and a polarized pair (X, L) such that

$$\bigoplus_{m \in \mathbb{Z}} V(\mathfrak{sl}_{r+1}, m\ell)|^{*}_{[C]} \cong \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^0(X, L^{\otimes m}).$$

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So for example, for $\mathfrak{sl}_2$, we know this works for ℓ divisible by 2.